LECTURE NOTES ON

STRUCTURAL ANALYSIS - I

Department of Civil Engineering

(B.Tech 4th Semester)

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- UDL longer than the span
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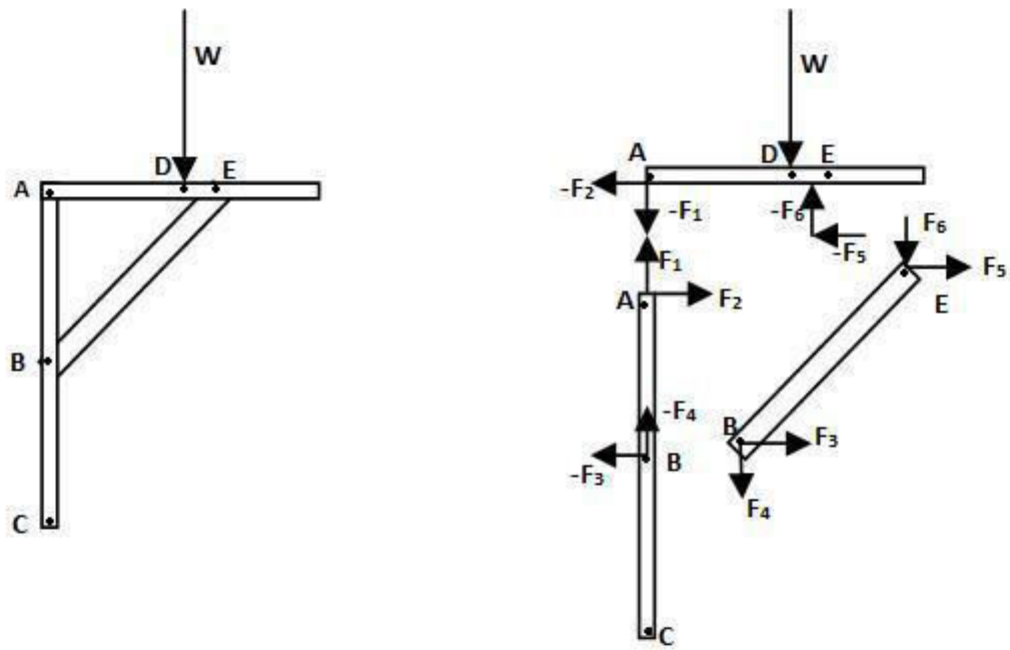
Chapter 1

Analysis of Perfect Frames

Introduction

Unlike the previous chapter in this unit we will be dealing with equilibrium of supporting structure. The structures may consist of several sections. They form the supporting structures of bridges, pillars, roofs etc. It is important to have a basic knowledge of this topic as it concerns with the safety and stability of a several important structures. We will be studying about the various internal forces responsible for keeping the structures together.

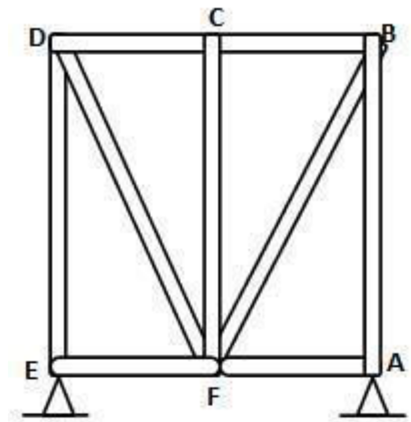
Following figure gives a basic idea of what we are going to study. The given figure is a normal diagram of a book shelf. The second figure shows the role of internal forces in maintaining the system equilibrium. Free body diagram of various components are shown. It is clear from the diagram that the forces of action and reaction between various parts are equal in magnitude and opposite in direction.



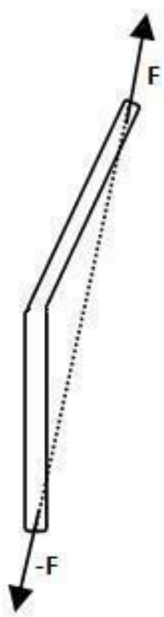
DEFINITION OF A TRUSS

Am truss is a network of straight slender members connected at the joints. Members are essentially connected at joints. The every member has force only at extremities. Further for equilibrium the forces in a member reduce to two force member. Thus no moments only two force member. In general trusses are designed to support. Trusses are designed to support weight only in its plane. Therefore trusses in general can be assumed to be 2-dimensional structures. Further in case weight of individual member is to be taken into consideration, half of them are to be distributed at each of the pinned ends.

Figure below shows a sample truss. There are nine individual members namely DE, DF, DC, BC, BF, BA, CF, EF, FA. Structure is 2-dimensional structure, supported by pin joints at A and E.



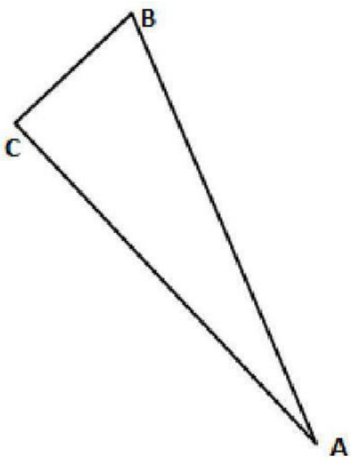
Sometime a member may be given of a shape like:

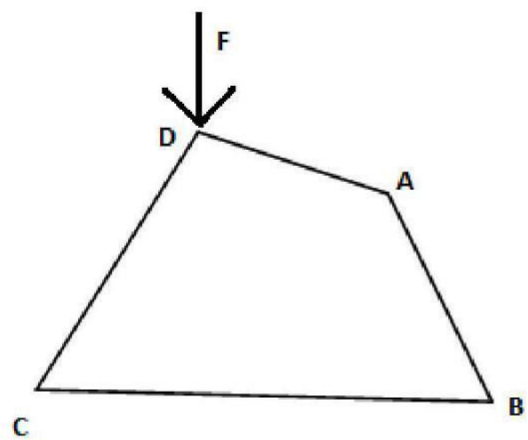


In such a case take the line the line joining their ends as the line of action of force.

ANALYSIS OF A TRUSS

A truss needs to be stable in all ways for security reasons. Simplest stable truss ABC is shown in figure.





The second diagram depicts that how instable the truss structure is. The truss ABCD can easily be deformed by application of the force F. Trusses constructed by adding triangles such as arms AC and CD to the above stable truss ABC are called simple trusses. No doubt simple trusses are rigid(stable). Further it is not always necessary that rigid trusses will necessary be simple.

Let m be no. of members and n be number of joints. For a truss

$m<2n-3$ deficiency of members, unstable ,fewer unknowns than equation.

$m= 2n-3$ uses all the members, statically determinate $m>2n-3$ excess member , statically indeterminate, more unknown than equation

METHOD OF JOINTS

In the following section we will consider about the various aspects of trusses. Distribution of forces, reactions forces at pins, tension and compression etc.

Step 1. Find the reaction at supporting pins using the force and the moment equations.

Step 2. Start with a pin, most preferably roller pin,wher there are 2 or less than two unknowns.

Step 3. Proceed in a similar way and try to find out force in different members one by one.

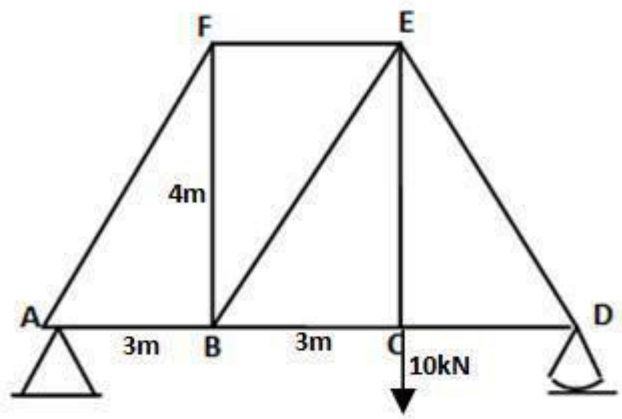
Step 4. Take care of while labelling forces on the members. Indicate compression and tension clearly. Step 5. Finally produce a completely labelled diagram.

Step 6.Try to identify the zero force members. It makes the problem simple.

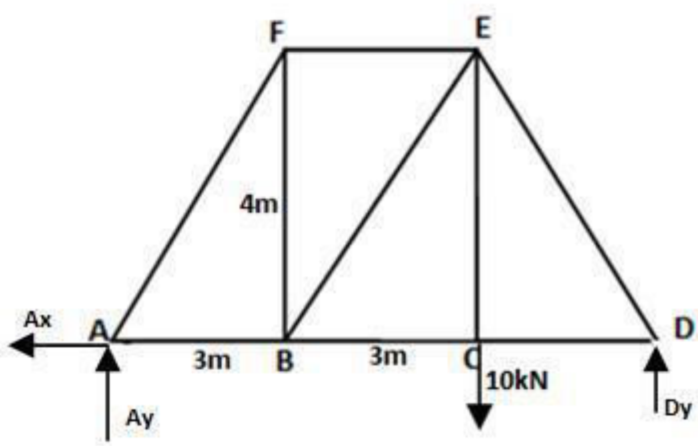
Compression Tension

Above shown are the conditions of compression or tension, deceided as per the direction of force applied by the pin joints to the members.

Example: Find the forces in the members AF, AB , CD, DE, EC and the reactin forces at A and D. CD = 3m.



Sol:



As per the type of joint the reaction forcces are shown below.

Clearly **Ax** = O (balancing forces horizontally)

Ay + Dy

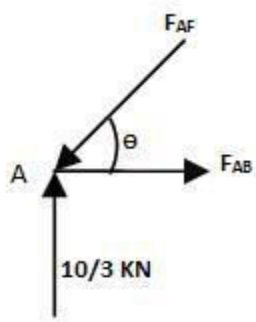
= 10 KN(balancing forces vertically)

taking

moment about D.

$\mathbf{M_D}$ = 10*3 – **Ay** *9 = 0 (zero for equlibrium) , therefore **Ay** = 10/3 KN , and **Dy** = 20/3 KN(10-10/3)

Now we have drawn the free body diagram of the pin A. We have assumed force at pin A due to the members in some direction.



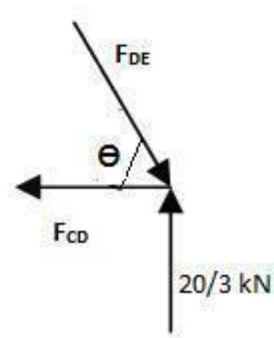
From the given data we can conclude tanθ= 4/3 , sinθ= 4/5 , cosθ = 3/5.

Balancing forces vertically. F_{AF} sin Θ = 10/3 . F_{AF} = 25/6 kN .

Balancig forces horizontally,F_{AB} = F_{AF} cos Θ = 2.5 kN .

Note the direction of the indicated forces are those applied be members to the pin. Force applied by pin onto the members will have the same magnitude but in opposite direction. Therefore we can easily state that member AF is in compression and member AB is in tension. Further each member is a two force member implying that it will exert the same amount of force to the pin on the other end but will be opposite in direction.

Now considering the joint D.

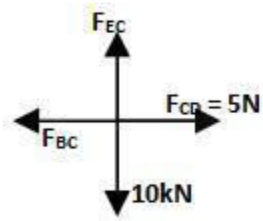


Balancing forces vertically. F_{DE} sin Θ = 20/3 . F_{AF} = 25/3 kN .

Balancig forces horizontally,$F_{CD} = F_{DE} \cos \Theta = 5$ kN .

Therefore we can easily state that member DE is in compression and member CD is in tension.

Now considering the joint C.



Balancing forces vertically. F_{EC} = 10kN .

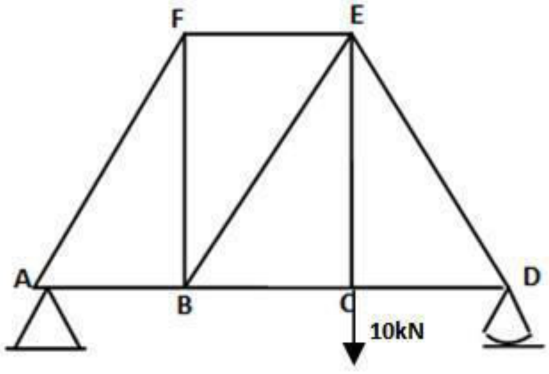
Balancing forces horizontally. F_{BC} = 5kN .

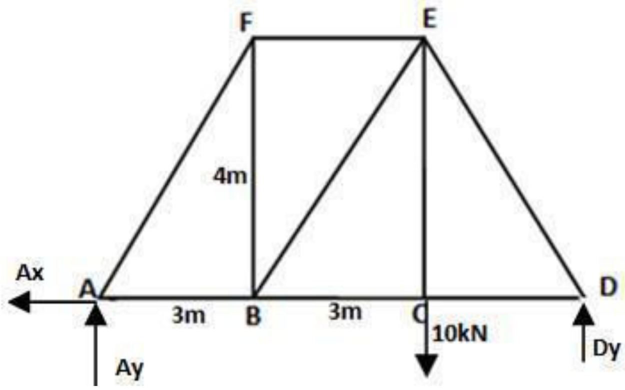
Therefore we can easily state both the members EC and BC are in tension.

In case in the above given problem 10kN was placed some where else, then ae per the FBD at joint C there would be no vertical force to balance F_{EC}. Hence force in EC would be zero. It is good to analyse the problem before hand and eliminate the zero force members, as they contribute nothing to the system.

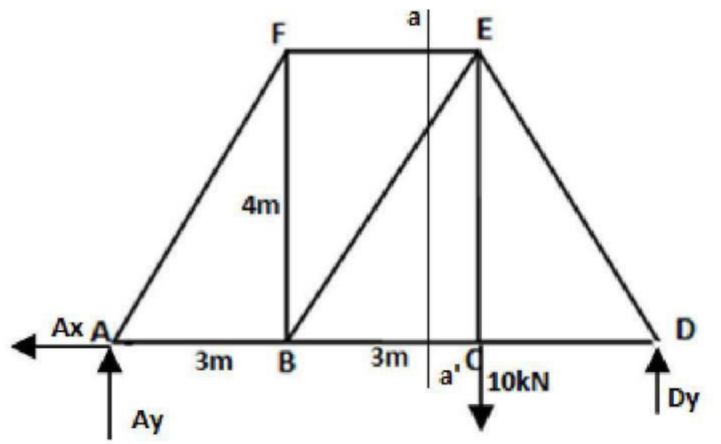
METHOD OF SECTIONS

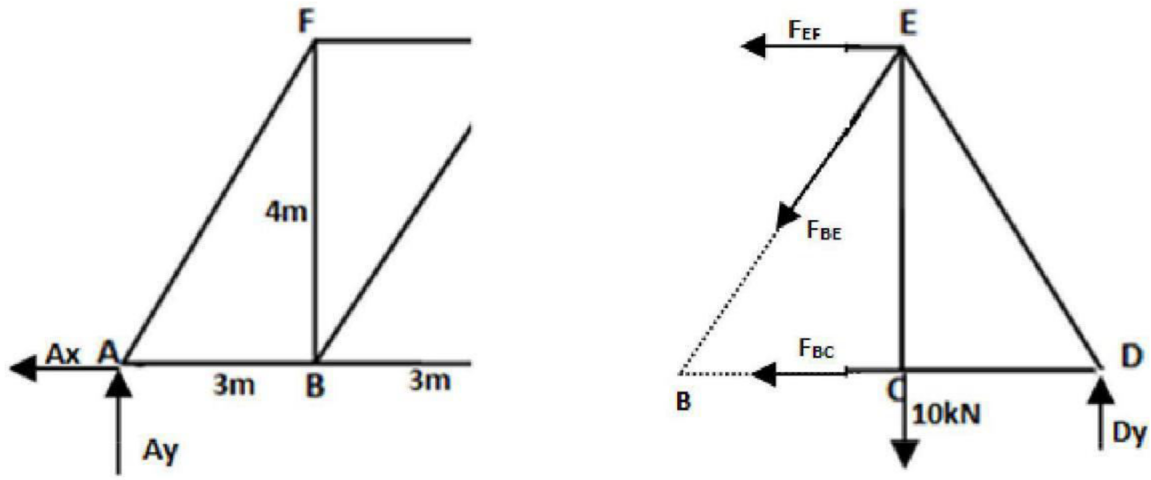
As the name suggests we need to consider an entire section instead of joints. When we need to find the force in all the members, method of joint is preferrable. For finding forces in few of the specific members method of joints is preferrable. Let us consider the same diagram as before.





We had been provided with the given system. We draw a axis aa’.



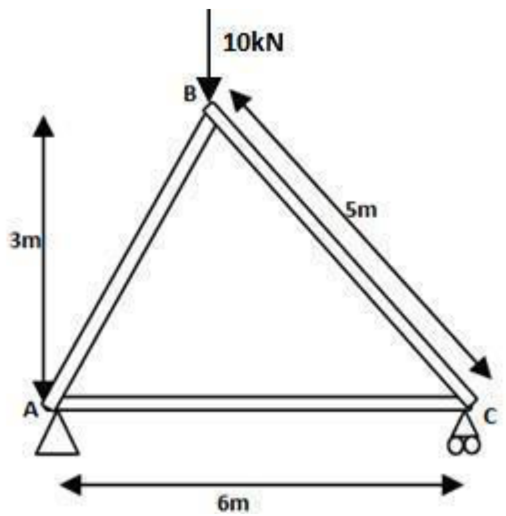


The axis should at max intersect three members. Then we separate the two sections apart. We can select any one of the part. We have just assumed he member to be in tension. We can find the reaction at supports. Now what we have done is divided the whole structure into two parts and taking into consideration various external reactions and member forces acting of one part.

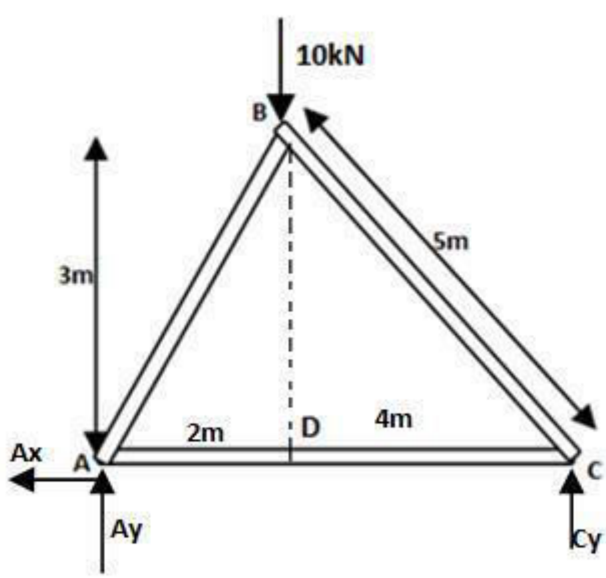
Suppose we have to find F_{EF}. It is sufficient to write the equation =0 (for equilibrium). To find F_{BC} it is M_B sufficient to write the equation of M_E =0. Similarly we can also the equations F_x =0 ,F_y =0 , for the use equilibrium of the section under consideration.

SOLVED PROBLEMS

1. Following is a simple truss. Find the forces in the all the members by method of joints.



Sol: we have the following given setup. By applying simple geometry we get AD=2m and CD=4m. We also assume a cetrtain reaction forces at the bottom. At C we have rollers, therefore we have reaction only in vertical direction.



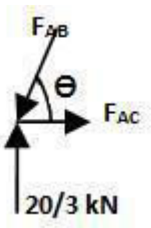
Clearly Ax=0 (balancing forces horizontally)

Ay+By=10kN

Taking M_A =0 we get : 10*2=Cy*6,implies **Cy**= 10/3 kN

Therefore **Ay** = 10-10/3 = 20/3 kN

Considering joint A

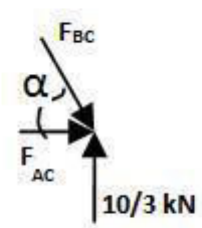


clearly tan ɵ = 1.5 therefore sin ɵ =0.83, cosɵ =0.554

Balancing forces vertically F_{AB} sinɵ = 20/3 , F_{AB} = 8.032 kN,compression

Balancing forces horizontally F_{AC}=F_{AC} cosɵ = 4.44kN , tension.

Considering joint C



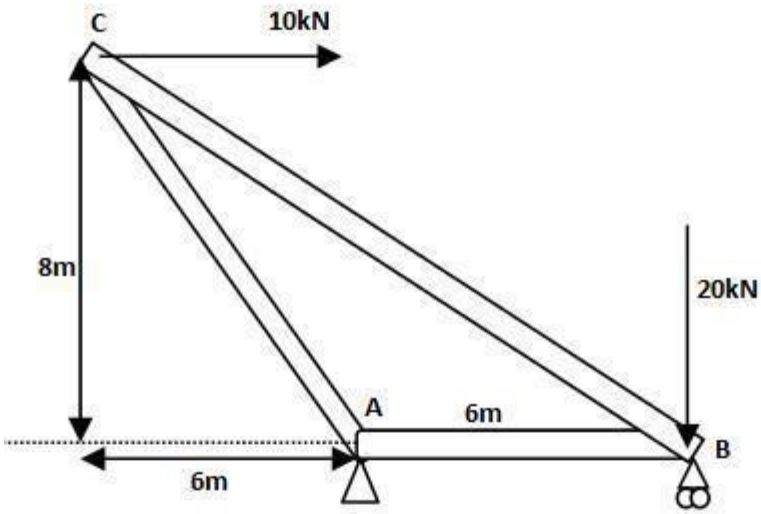
clearly tan α = 0.75 therefore sin α =0.6, cosα =0.8 Balancing forces vertically F_{BC} sinα = 10/3 , F_{BC} = 5.55 kN,compression

Balancing forces horizontally F_{AC}=-F_{AC} cosɵ = 4.44kN , tension(correctly verified).

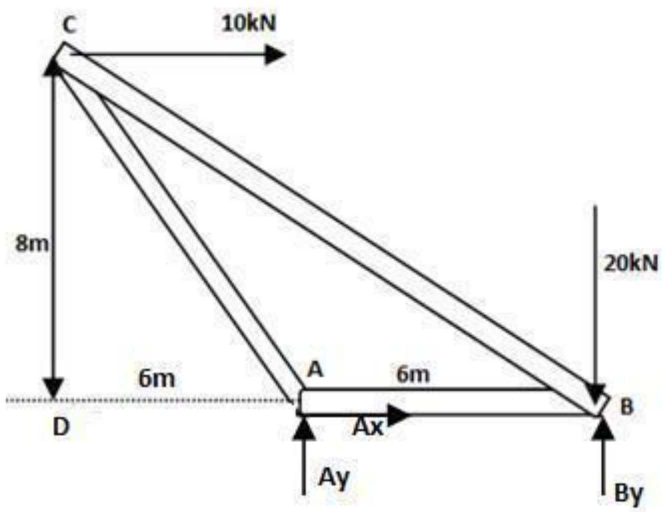
F_{AB} = 8.032kN(compression), F_{BC} = 5.55 kN,(compression)

F_A= 4.44kN , (tension)

2. Find the reaction components at A and B. Also find the forces in each individual member, specify compression or tension.



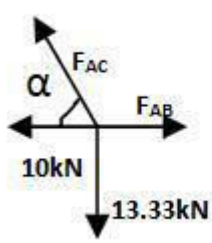
sol: let us assume the reacton forces as:



Ax = -10kN(towards the left), Ay + By = 20kN .

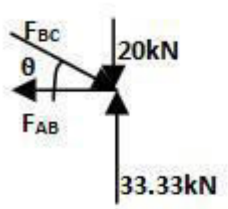
M_A = 0 gives : 10*8 = (By-20)*6 , By = 33.33kN , therefore Ay = -13.33kN (actualy downwards).

Taking point A:



cos α = 0.6,sin α = 0.8 .

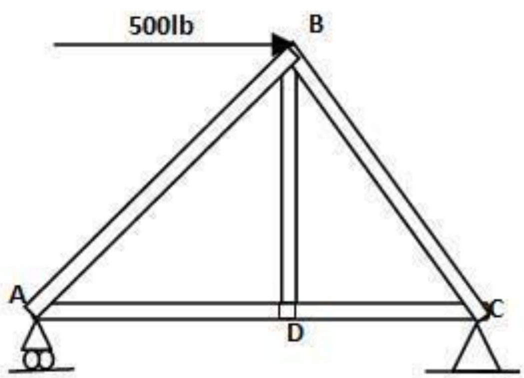
F_{AC} sinα = 13.33 ,**F_{AC} = 16.66kN**.(tension) and **F_{AB}=** 10+F_{AC}cos α =**20kN(**tension**).**



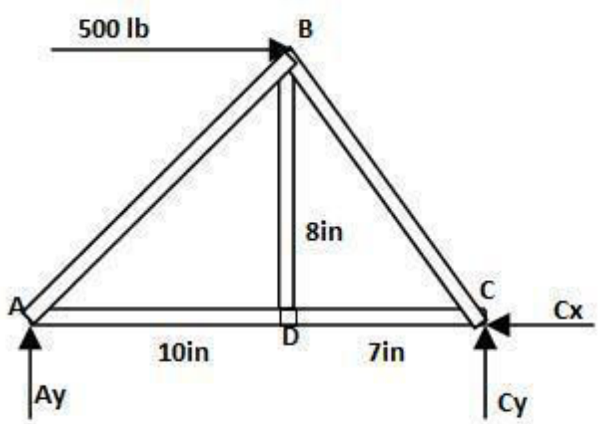
cos ɵ = 0.83,sin ɵ = 0.55 .

F_{BC} sinɵ = 13.33 ,**F_{BC} = 24.23kN**.(compression) and **F_{AB}=** F_{BC}cos α =**20kN(**tension**).(**hence verified**).**

3. Find the reaction components at A and C. Also find the forces in each individual member, specify compression or tension. Given AD=10in ,DC=7in , BD=8in.

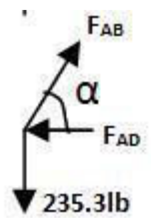


Sol: let us assume the reacton forces as:



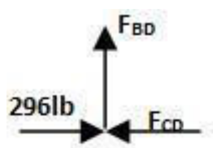
Cx = 500lb(towards the left), Ay + Cy = 0 .or, Ay = -Cy

M_A = 0 gives : Cy*17 = 500*8 , Cy = 235.3lb , therefore Ay = -235.3kN (actualy downwards). Taking point A:



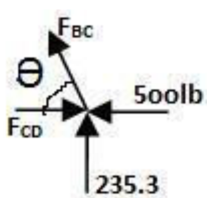
cos α = 0.78,sin α = 0.62 .

F_{AB} sinα = 235.3 ,$\mathbf{F_{AB}}$ = **380lb**.(tension) and $\mathbf{F_{AD}}$**=** F_{AB}cos α =**296lb(**compression**).** Taking point D:



F_{CD} = 296lb(compression), F_{BD} =0 lb (zero force member, should have been removed in the beginning itself)

Taking point C:

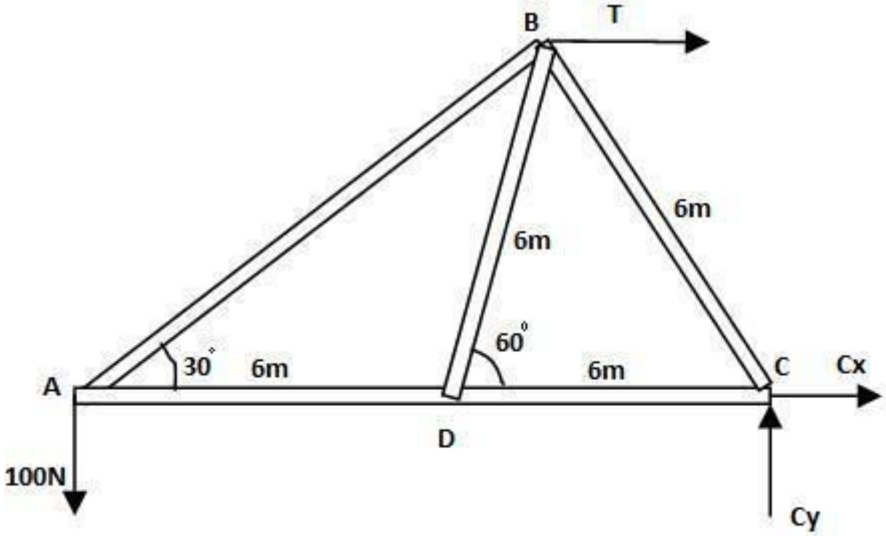


cos ɵ = 0.65,sin ɵ = 0.75 .

F_{BC} sinɵ = -253.3,$\mathbf{F_{BC}}$ **= -337.73lb**.(compression). F_{CD} = 500+F_{BC} cos ɵ = 296lb (compression**).**

4. Find the forces in the members and the reaction forces. All relevent details are provided below.

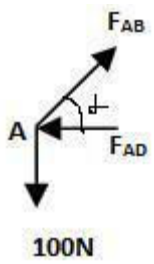
Sol: From the given figure we can conclude that triangle BCD is equilateral and triangle ABD is isoceles.



clearly **Cy = 100N**(balancing forces horizontally)

M_c = 0 gives , T * $\frac{6\sqrt{3}}{2}$ = 100 * 12 or **T = 230.9 N** , and **Cx** = -T = **-230.9 N**(towards the left).

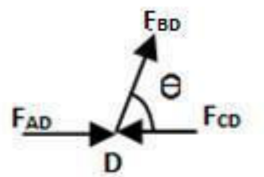
Considering point A : ___________



$\cos \alpha = 0.86, \sin \alpha = 0.5$.

$F_{AB} \sin\alpha = 100$,**F_{AB} = 200N**(tension). $F_{AD} = F_{AB} \cos \alpha = 172N$ (compression**).**

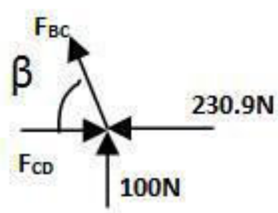
Considering point D : _______________



$\theta = 60°$

$F_{BD} \sin\theta = 0$,**F_{BD} = 0 N(**zero force member**)**. $F_{CD} = F_{AD} = 172N$ (compression**).**

Considering point C :_______________

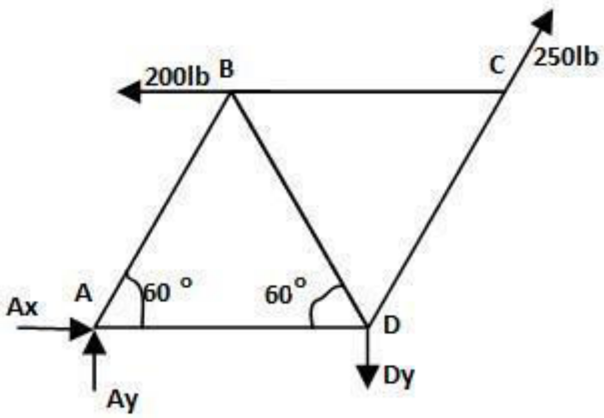


$\alpha = 0.5, \sin \alpha = 0.86$.

$F_{BC} \sin\beta = -100$, **F_{BC} = -116.27N** (compression). $F_{CD} = 230.9 + F_{BC} \cos \beta = 172N$ (compression**)(**verified**).**

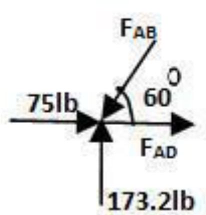
5. Find the forces in all the members of the of the following structure. Let the tension in the string be 250lb. would it be possible to find the to solve the problem if the tension in the string was unknown.

Sol:



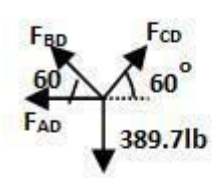
Balancing forces horizontally we get : Ax + 250 cos60 = 200 or **Ax = 75lb.**
Balancing forces vertically we get : Dy-Ay= 250 sin 60 = 216.50 ,
M_D =0, gives Ay * 12 = 200 * $\frac{12\sqrt{3}}{2}$
or **Ay = 173.2 lb** and **Dy = 389.7 lb**

Lets consider the point A**:**



cos 60 = 0.5,sin 60= 0.86 .
F_{AB} sin60 = 173.2,$\mathbf{F_{AB}}$ **= 201.39lb**(compression). F_{AD} =F_{AB} cos 60-75 = 25.69lb(tension**).**

Lets consider the point D**:**

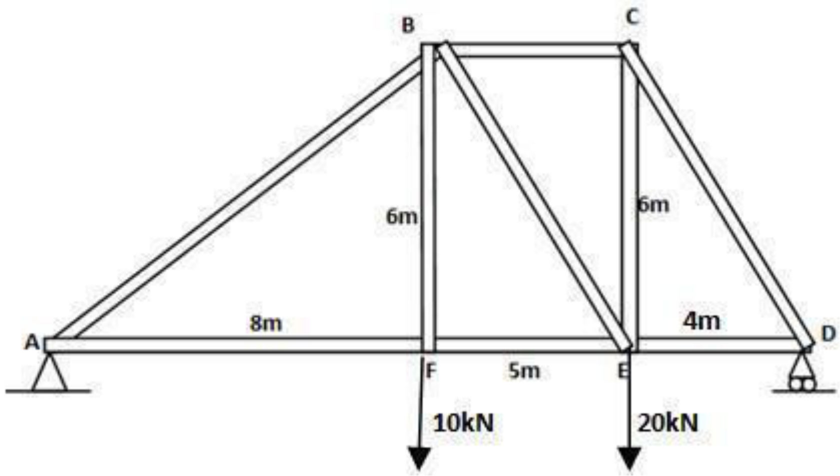


we just found out that F_{AD} **=** 25.69lb
balancing forces vertically we get:(F_{BD}**+$\mathbf{F_{CD}}$**)sin60 = 398.7 ,or, F_{BD}+F_{CD} = 450 balancing forces horizontally we get:(F_{BD}**-$\mathbf{F_{CD}}$**)cos60 = -25.69 ,or, F_{BD} -F_{CD}= -51.38 or **$\mathbf{F_{BD}}$= 199.31lb(**tension**)** and **$\mathbf{F_{CD}}$= 250.69lb(**tension**).**

At joint C we will find that F_{CD} and T are almost equal and will cancel off. Therefore BC becomes a zero force member.

NO. The problem cannot be solved if the tension wasn't given, as it would introduce four unknowns in the systems. More than three unknowns will become difficulat to handel with just three equations.

6. Using method of sections find forces in the members BC,EF .



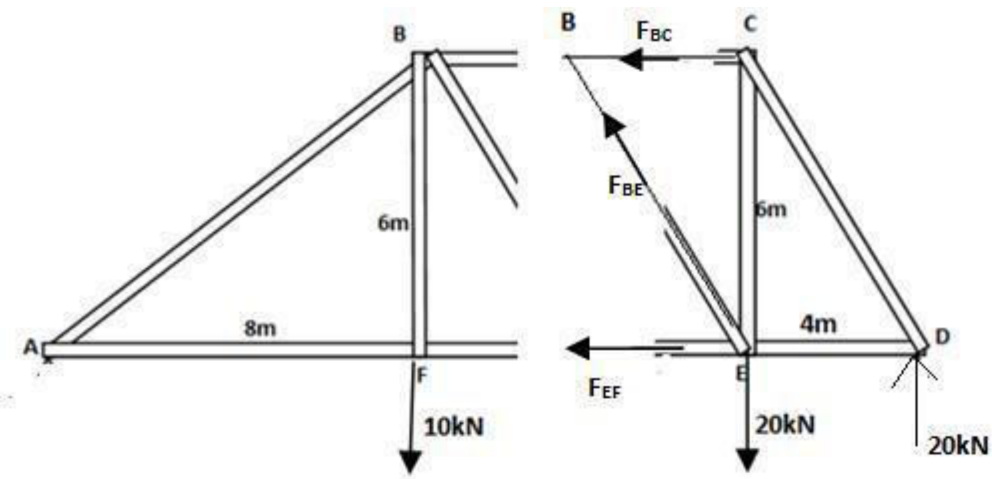
Sol : let us first figure out the reaction force at D. Let the reaction force at D be **Dy** in vertically upward direction.

Now we write the equation of M_A =0, Dy *17 = 10*8 + 20*13 .

Or **Dy = 20Nm**

To proceed by the method of sections we need to deceide an axis. Let's take an axis aa' as shown in the figure below.

After splitting into two sections we get:

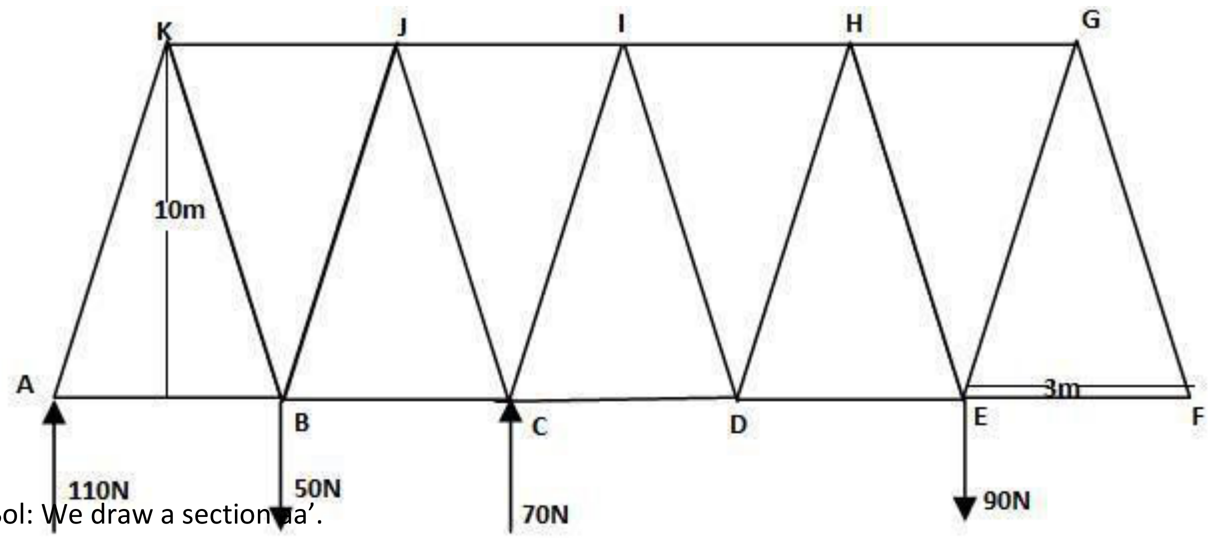


We will take the right part into consideration. We have just assumed the direction of forces, they can turn out to be opposite.

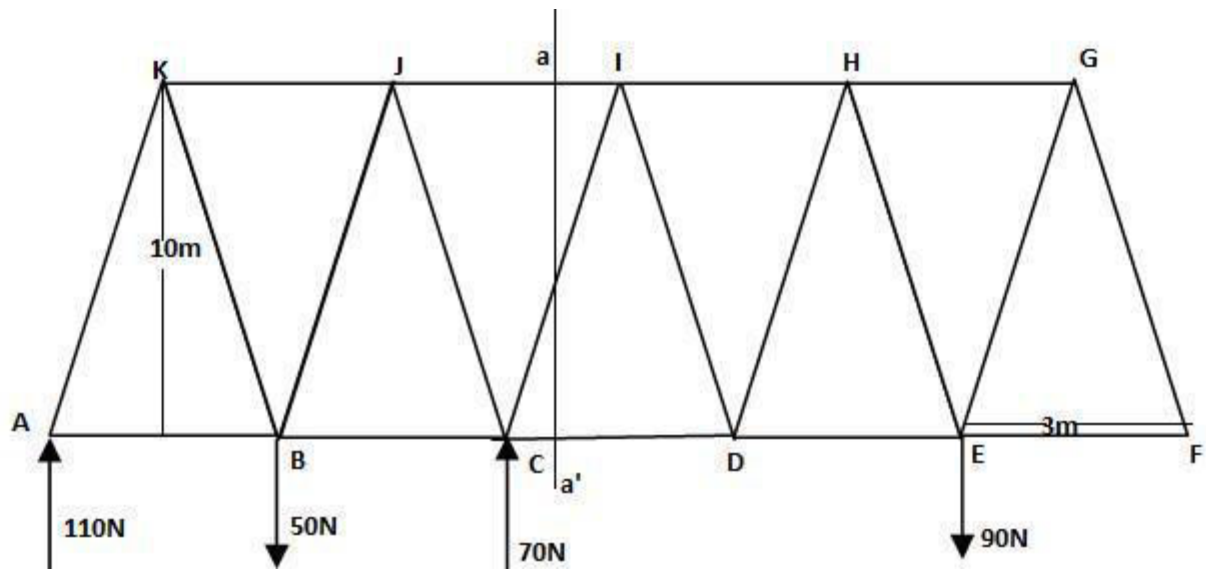
M_E = 0 , gives F_{BC} * 6 = - 20*4 , or F_{BC} = -13.33N (member is in compression)

M_C = 0 , gives F_{EF} * 6 = 20*4 , or F_{EF} = 13.33N (member is in tension as assumed).

7. Using method of sections find forces in the member JI,CD,CI. All triangles are congruent.



Sol: We draw a section aa'.



After splliting the section we have:(we take the left part into consideration)

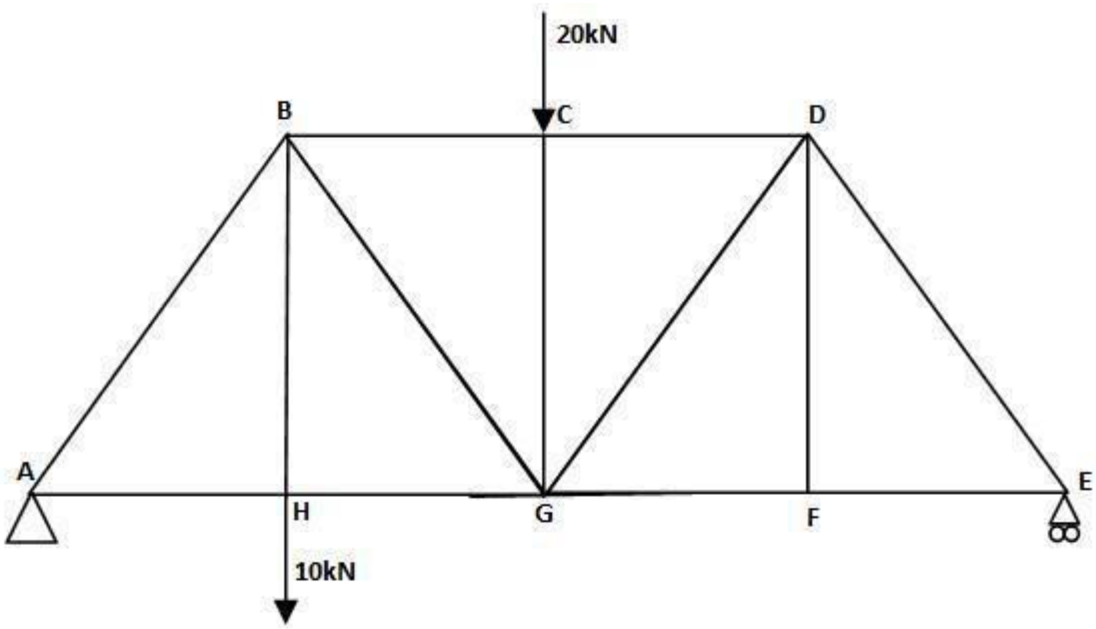
$\tan\alpha = 6.66$, $\sin \alpha = 0.98$, $\cos \alpha = 0.148$

M_C =0,$50*3–110*6-F_{JI} *10=0$,**F_{JI} =-51N.**

M_I = 0, $50 * 4.5 – 110 *7.5 – 70*1.5 + F_{CD} * 10 = 0$, **F_{CD} = 70.5N.**

Balancing forces horizontally: $F_{CI} \cos\alpha = -F_{CD} - F_{JI} = -70.5 + 51 = 19.5$ N, or, **F_{JI}** = **131.75N**

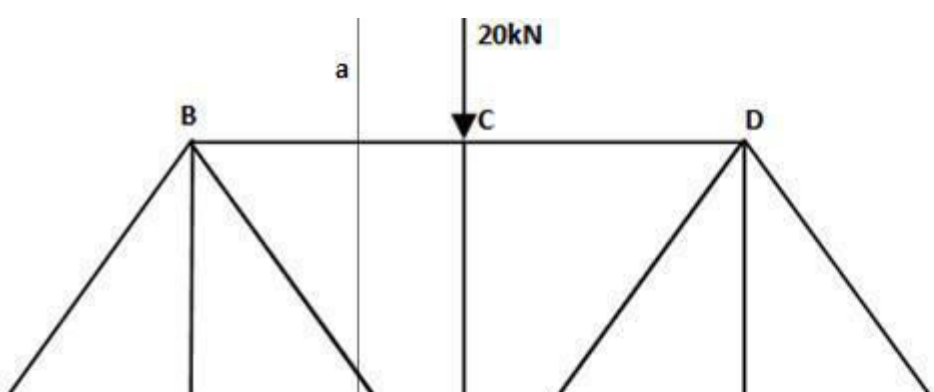
8. Find the force in members CG, FG, BG,BC. Use method of section to compute the result. Indicate the zero force member. BH = 4m , AH=HG=GF=FE = 3m,



Sol: As clearly visible DF is a zero force member.

Ay + Ey = (20 -10)kN = 10 kN.

Ax=0(balancing forces horizontally) .



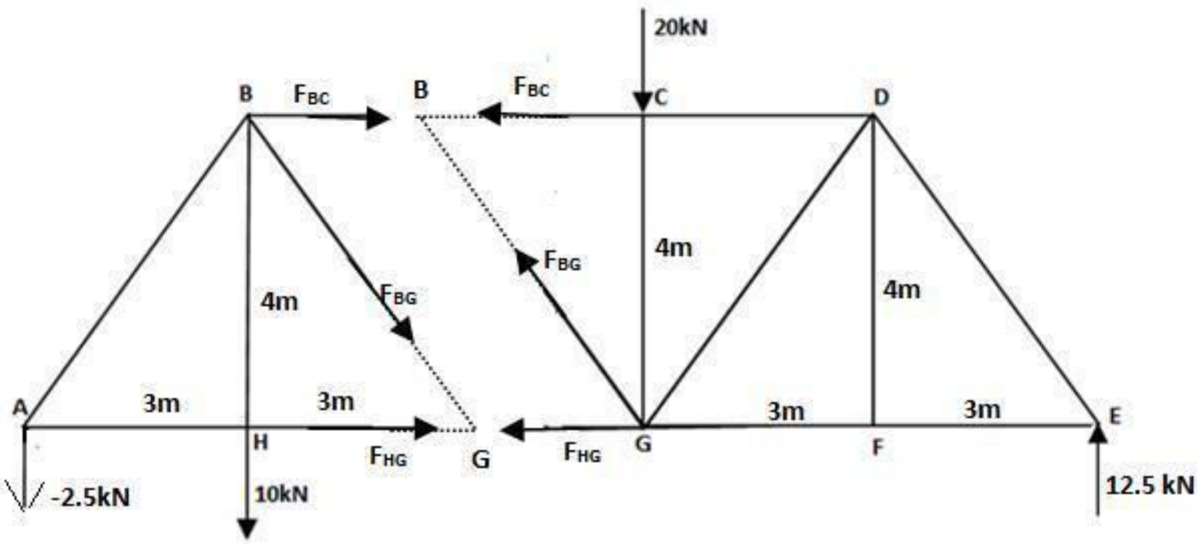
M_A = 0; Ey * 12 = 10*3 + 20 *6 ;

Ey = 12.5

N , therefore **Ay = - 2.5kN**.

We draw the section aa'

and split the diagram into two parts.



taking the left part into consideration.

M_B= 0 ; F_{HG} * 4 = - 2.5 * 3 , **F_{HG} = -1.875 kN**

M_G=0;F_{BC} *4=10*3,

F_{BC} = 7.5 kN.

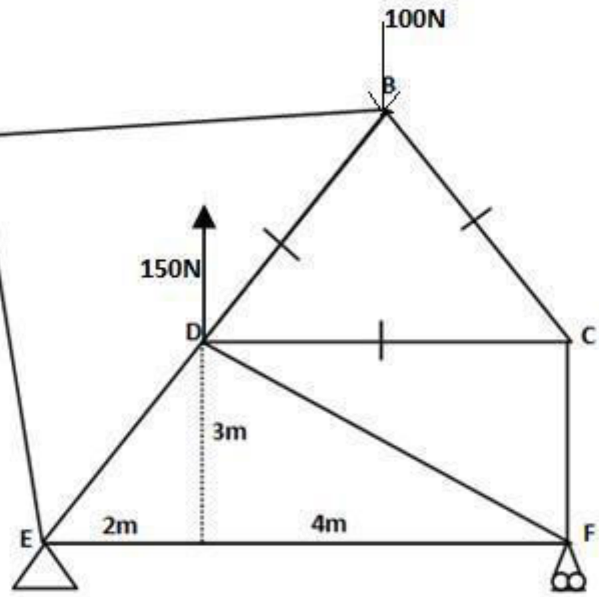
Bala

ncing forces horizontally : - 1.875 + 7.5 + F_{BG} cos α =0 , or , **F_{BG} = - 9.375N.**

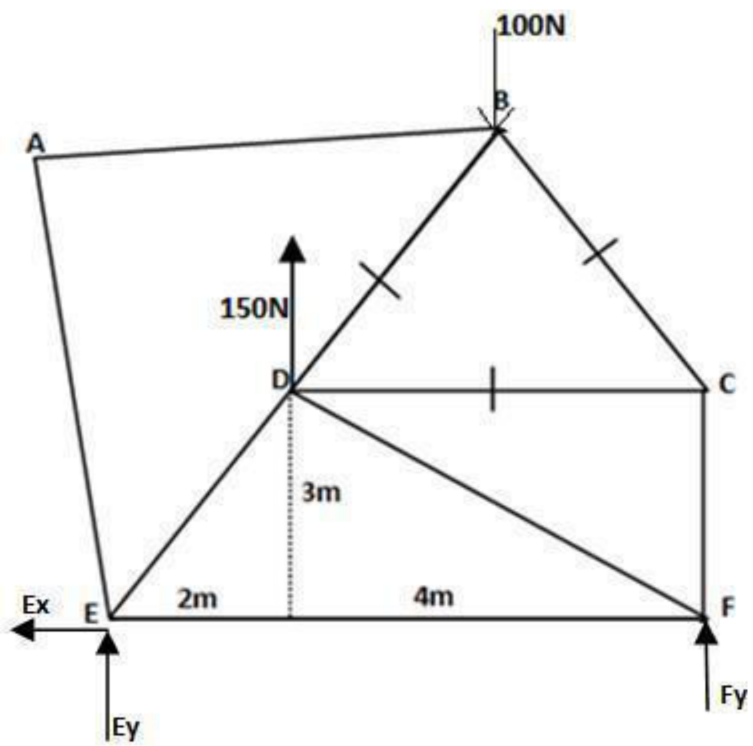
Considering point C: 20kN clearly F_G = 20kN , since there are no vertical components of other forces.

F_{CG}

9. Find the forces in members AD,DC,EF,CF,BD,BCof the following given truss.



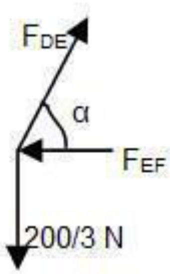
Sol: Members AE and AB are zero force members therefore they can be eliminated from the system.



Ex=0 N

Ey+Fy = - 50N ; M_E = 0 = 150* 2 + Fy* 6 = 100 *4 : **Fy = 50/3 N** ,and also ,**Ey = -200/3 N .**

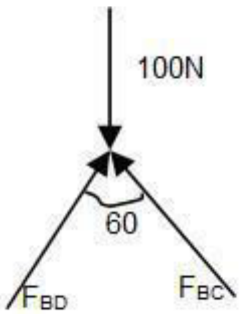
<u>Considering pin E:</u>



-tan α = 1.5 , cos α =0.55 ,sinα = 0.83 ;

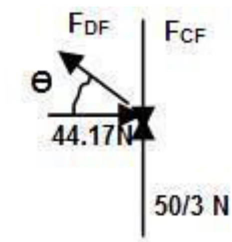
F_{DE} = 200/(3sin α) = 80.32N(compression) , F_{EF} = F_{DE} cosα = 44.17 N(tension).

Considering pin B:



due to the symmetrical setup : F_{BD} = F_{BC} and 2F_{BD}cos 30=2F_{BC}cos 30 = 100 F_{BD}= F_{BC} = 57.73 N (both compression)

Considering pin F:

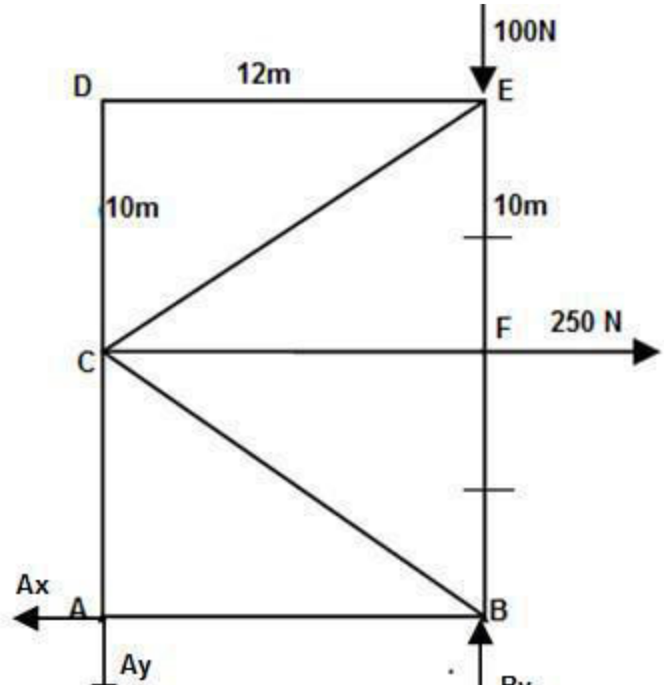


cos θ = 0.8 , sin θ = 0.6

F_{DF} =44.17 **/ cos θ = 55.21 N(**tension**) , F_{CF} =** 50/3 - F_{DF} sinθ = - 16.46N(compression) 10.

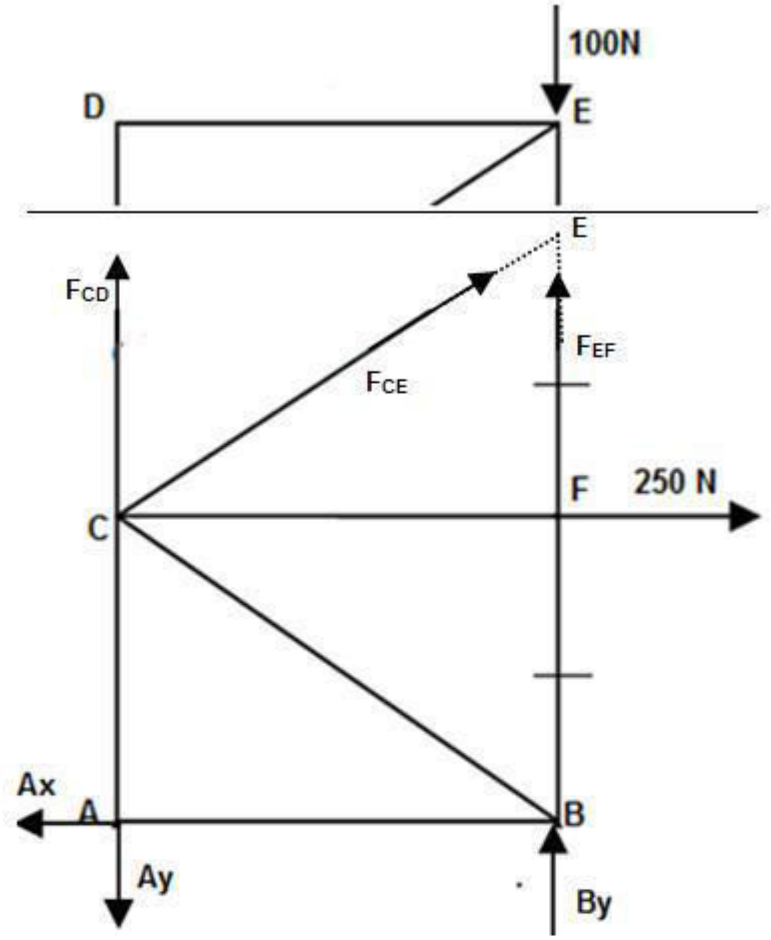
Find the reaction forces and force in CD,CE and EF using method of sections

Sol: We have the following diagram.



Ax = 250 N , By-Ay = 100 N, $\mathbf{M_A}$ = 0 = By * 12 = 250 * 10 + 100*12 ; or **By = 308 N and Ay =208N.**

We divide the section as beloW



Considering the lower part $M_C = 0$; $(F_{EF}+308)*12 = 250*10$, $F_{EF} = -100$ N(compression)

Considering the upper section : $M_E = 0$; $F_{CD}=0$

At the joint E there is no horizontal force to counter balance the horizontal component of F_{CE}. Therefore $F_{CE} = 0$;

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Chapter 2

Energy Theorems and Three Hinged Arches

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1. INTRODUCTION

When an elastic body is deformed, work is done. The energy used up is stored in the body as strain energy and it may be regained by allowing the body to relax. The best example of this is a clockwork device which stores strain energy and then gives it up.

We will examine strain energy associated with the most common forms of stress encountered in structures and use it to calculate the deflection of structures. Strain energy is usually given the symbol U.

2. STRAIN ENERGY DUE TO DIRECT STRESS.

Consider a bar of length L and cross sectional area A. If a tensile force is applied it stretches and the graph of force v extension is usually a straight line as shown. When the force reaches a value of F and corresponding extension x, the work done (W) is the area under the graph. Hence **W = Fx/2.** (The same as the average force x extension).

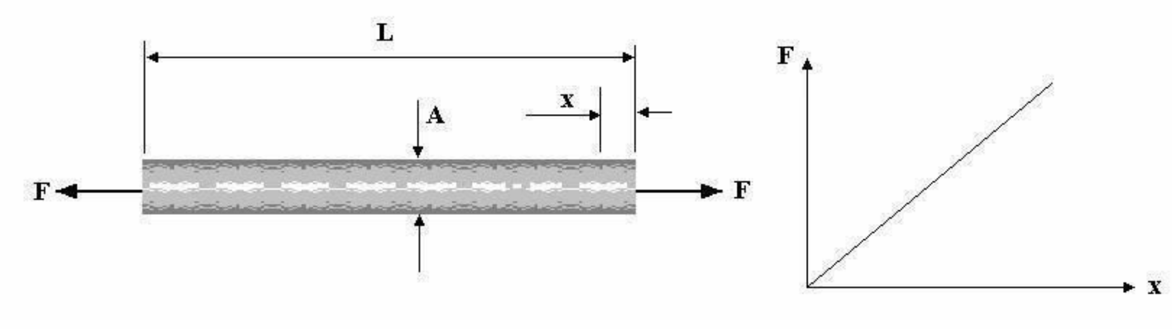


Figure 1

Since the work done is the energy used up, this is now stored in the material as strain energy hence **U = Fx/2**

The stress in the bar is = F/A hence F = A
The strain in the bar is = x/δ hence x = δ

For an elastic material up to the limit of **proportionality, / = E (The modulus of elasticity) hence = /E**

Substituting we find ← = A δ/2 = ^{2} AL/2E

The volume of the bar is A L so **U = (∠ /2E) x volume of the bar**

EX: A steel rod has a square cross section 10 mm x 10 mm and a length of 2 m. Calculate the strain energy when a stress of 400 MPa is produced by stretching it. Take E = 200 GPa

SOLUTION

$A = 10 \times 10 = 100\ mm^2$ or $100 \times 10^{-6}\ m^2$. $V = AL = 100 \times 10^{-6} \times 2 = 200 \times 10^{-6}\ m^3$.

$= \mathbf{400 \times 10^6}\ N/m^2$ and $E = 200 \times 109\ N/m^2$

$$U = \frac{{}^2}{2E} \times Volume = \frac{(400 \times 10^6)^2}{2 \times 200 \times 10^9} x200x10^{-6} = 80\ Joules$$

3. STRAIN ENERGY DUE TO PURE SHEAR STRESS

Consider a rectangular element subjected to pure shear so that it deforms as shown. The height is h and plan area A. It is distorted a distance x by a shear force F. The graph of Force plotted against x is normally a straight line so long as the material remains elastic. The work done is the area under the F - x graph so **W = Fx/2**

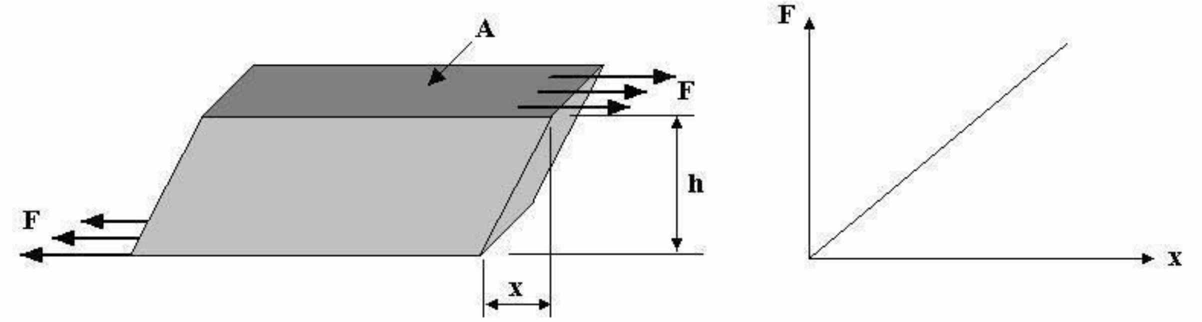


Figure 2

The work done is the strain energy stored hence **U = Fx/2**

The shear stress is = F/A hence F = A

The shear strain is γ = x/h hence x = γh

Note that since x is **very small it is the same length as an arc of radius h and angle** γ. **It** follows that the shear strain is the angle through which the element is distorted.

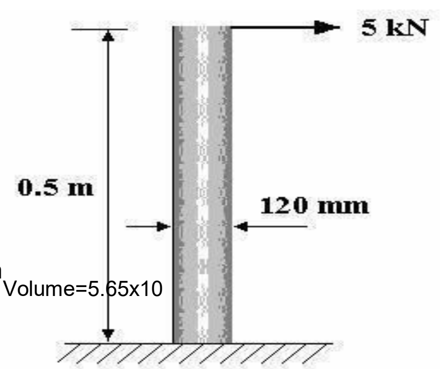
For an elastic material /γ = G (The modulus of Rigidity) hence γ= /G

Substituting we find U = **Aγh/2** = ^{2}Ah/2G

The volume of the element is A h so ← = (^{2}/2G) x volume

Pure shear does not often occur in structures and the numerical values are very small compared to that due to other forms of loading so it is often (but not always) ignored.

EX: Calculate the strain energy due to the shear strain in the structure shown. Take G = 90GPa



SOLUTION

$A = \pi d^2/4 = \pi \times 0.12^2/4 = 11.31 \times 10^{-3}\ m^2$

= **F/A = 5000/ 11.31 x 10^{-3}** = 56.55

^{2} Volume = A h = 11.31 x 10^{-3} x 0.5

2^{-3} m$^{3N/m}$ Volume=5.65x10

← = (/2G) x volume 15

$U = \{(56.55)^2/(2 \times 90 \times 10^9)\} \times 5.65 \times$

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10^{-3} $U = 100.5 \text{ x } 10^{-12}$ Joules

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Note that the structure is also subject to bending. The strain energy due to bending is covered later.

Figure 3

4. STRAIN ENERGY DUE TO TORSION

Consider a round bar being twisted by a torque T. A line along the length rotates through **angle γ and the corresponding radial line on the face rotates angle θ. γ is the shear strain on** the surface at radius R.

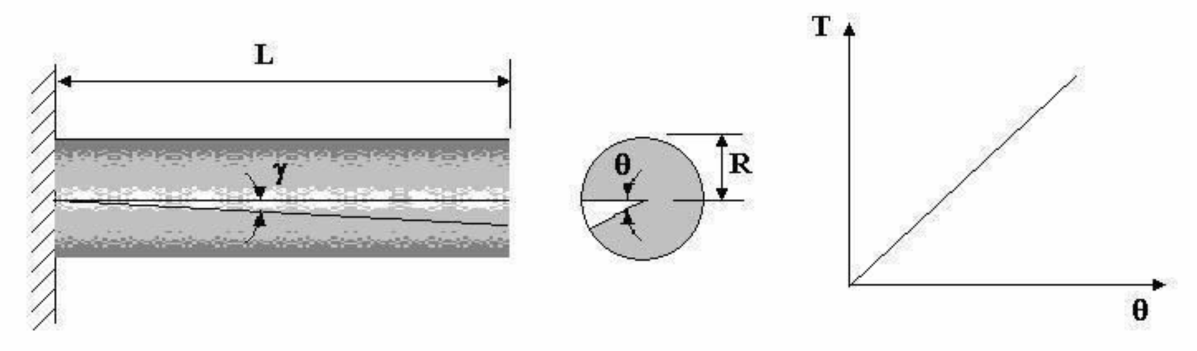

Figure 4

The relationship between torque T and angle of twist **θ is normally a straight line. The work** done is the area under the torque-**angle graph. For a given pair of values → = Tθ/2**

The strain energy stored is equal to the work done hence ← =
Tθ/2 from **the theory of torsion (not covered here) θ = Tδ/GJ**

G is the **modulus of rigidity and J is the polar second moment of area. J = πR^4** /2 for a solid circle.

Substitute θ = Tδ/GJ and we get ← = T^2 L/2GJ

Also from torsion theory T = J/R where is maximum shear stress on the surface.

Substituting for T we get the following.

← = **(J/R)^2 /2GJ =** ^{2}JL/2GR^2 **Substitute J = πR^4** /2

← = ^{2} **πR^4** L/4GR^2 = ^{2}**πR^2** L/4G

The volume of the bar is Aδ = πR^2 L so it follows that:

U = (^{2}/4G) x volume of the bar. (is the maximum shear stress on the surface)

EX: A solid bar is 20 mm diameter and 0.8 m long. It is subjected to a torque of 30 Nm. Calculate the maximum shear stress and the strain energy stored. Take G = 90GPa

SOLUTION

R = 10 mm = 0.01 m L = 0.8 m

A = **π**R^2 = **π** x 0.01^2 = 314.16 x 10^{-6} m^2

Volume of bar = AL = $314.16 \times 10^{-6} \times 0.8 = 251.3 \times 10^{-6}\ m^3$

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$J = \pi R^4/2 = \pi\,(0.01)^4/2 = 15.7 \times 10^{-9}\ m^4$

$\tau = TR/J = 30 \times 0.01/15.7 \times 10^{-9} = 19.1 \times 10^{6}\ N/m^2$

$U = (\tau^2/4G) \times$ volume of the bar $= \{(19.1 \times 10^{6})^2/(4 \times 90 \times 10^{9})\} \times 251.3 \times 10^{-6}$

U = 0.255 Joules

A helical spring is constructed by taking a wire of diameter d and length L and coiling it into a helix of mean diameter D with n coils. Show that the stiffness of the helical spring shown below is given by the formula $F/y = Gd^4/8nD^3$

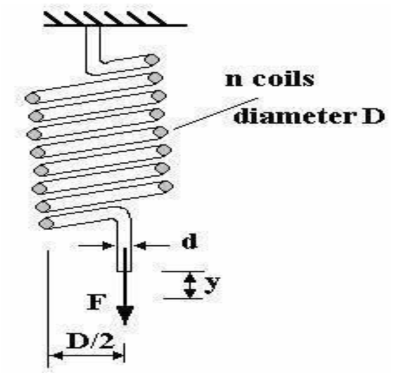

Figure 5

SOLUTION

When a force F is applied to the end it deflects down by a distance y. Looking at the bottom coil, it can be seen that a torque T = FD/2 is twisting the cross section of the wire. This torsion is transmitted throughout the entire length of the wire.

Starting with the strain energy due to torsion we have:

$U = (\tau^2/4G) \times$ volume of the bar

And substituting V = AL and $\tau = Td/2J$

$$U = \frac{\left(\frac{Td}{2J}\right)^2}{4G} \times AL = \frac{T^2 d^2}{16GJ^2} \times AL = \frac{T^2 d^2}{16GJ^2} \times \frac{\pi d^2}{4} \times L = U = \frac{T^2 \pi d^4}{64GJ^2} \times L$$

Since $J = \frac{\pi d^4}{32}$ this reduces to $U = \frac{T^2}{2GJ} \times L$

The work done by a force F is ½ Fy. Equating to U we get:

$$\frac{Fy}{2} = \frac{T^2}{2GJ} \times L = \frac{F^2 (D/2)^2}{2GJ} = \frac{F^2 D^2}{8GJ} \times L$$

$$\frac{F}{y} = \frac{4GJ}{LD^2} = \frac{4G\frac{\pi d^4}{32}}{LD^2} = \frac{Gd^4}{8LD^2}$$

$\frac{F}{y} = \frac{G\pi d^4}{8(n\pi D)D^2}$ and substitute $L = n\pi D$

$$\frac{F}{y} = \frac{Gd^4}{8nD^3}$$

This is the well known equation for the stiffness of a helical spring and the same

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formula may be derived by other methods.

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This is the well known equation for the stiffness of a helical spring and the same formula may be derived by other methods.

5. STRAIN ENERGY DUE TO BENDING.

The strain energy produced by bending is usually large in comparison to the other forms. When a beam bends, layers on one side of the neutral axis are stretched and on the other side they are compressed. In both cases, this represents stored strain energy. Consider a point on a beam where the bending moment is M.

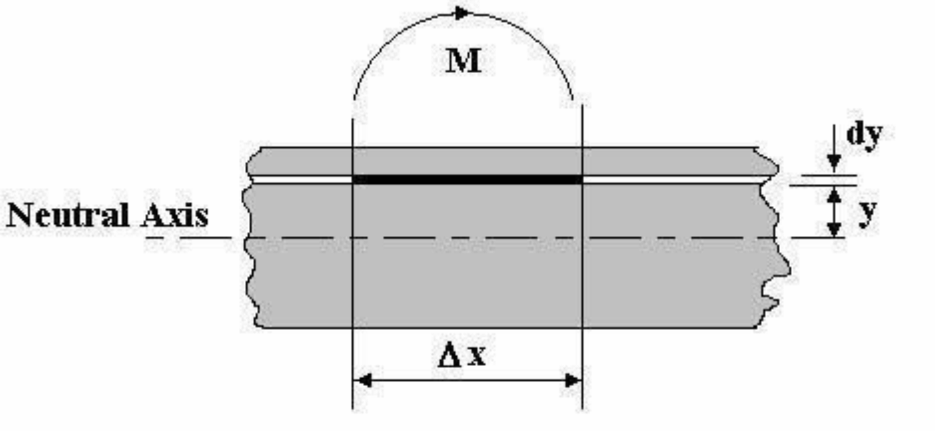

Figure 6

Now consider an elementary layer within the material of length Δx and thickness dy at distance y from the neutral axis. The cross sectional area of the strip is dA.

The bending stress is zero on the neutral axis and increases with distance y. This is tensile on one side and compressive on the other. If the beam has a uniform section the stress distribution is as shown.

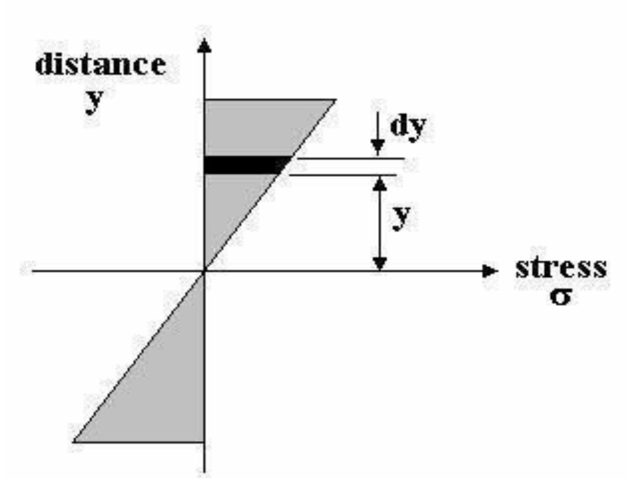

Figure 7

Each elementary layer has a direct stress ( ) on it and the strain energy stored has been shown to be U = (^{2}/2E) x volume (in section 2)

The volume of the strip is Δx dA

The strain energy in the strip is part of the total so du = (^{2}/2E)Δx dA

From bending theory (not covered here) we have = My/I where I is the second moment of area.

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Substituting for we get

$$du = \frac{(My/I)^2}{2E} \Delta x\, dA \text{ and in the limit as } \Delta x \to dx$$

$$du = \frac{(My/I)^2}{2E} dx\, dA = \frac{M^2}{2EI^2} y^2 dA\, dx$$

$du = \{(My/I)^2/2E\}\Delta x\, dA$

The strain energy stored in an element of length dx is then

$u = \frac{M^2}{2EI^2} dx \int y^2 dA$ and by definition $I = \int y^2 dA$ so this simplifies to

$$u = \frac{M^2 dx}{2EI}$$

In order to solve the strain energy stored in a finite length, we must integrate with respect to x.

For a length of beam the total strain energy is $U = \frac{1}{2EI}\int M^2 dx$

The problem however, is that M varies with x and M as a function of x has to be substituted.

EX: Determine the strain energy in the cantilever beam shown. The flexural stiffness EI is 200 kNm^2.

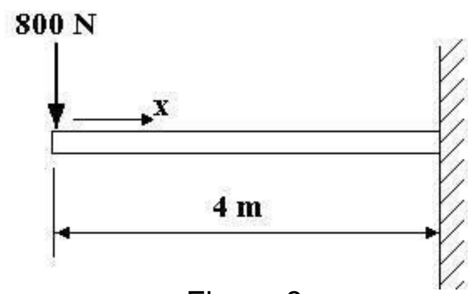


Figure 8

SOLUTION

This is a bending problem so $U = \frac{1}{2EI}\int M^2 dx$

The beam is a simple cantilever so the bending moment at any distance x from the end is simply M = -800 x (The minus sign for hogging makes no difference since it will be squared)

$$U = \frac{1}{2EI}\int_0^4 M^2 dx = \frac{1}{2EI}\int_0^4 (-800x)^2 dx = \frac{1}{2EI}\int_0^4 640000\, x^2 dx$$

$$U = \frac{640000}{2EI}\int_0^4 x^2 dx = \frac{640000}{2EI}\left[\frac{x^3}{3}\right]_0^4$$

$2 \times 2 \times 10^{5}$ $2 \times 2 \times 10^{5}$ 3

$$U = \frac{640000}{2 \times 2 \times 10}\left[\frac{4^3}{3} - 0\right] = 34.13 \text{ Joules}$$

6. DEFLECTION

The deflection of simple structures may be found by equating the strain energy to the work done. This is covered in detail later but for the simple cantilever beam it can be demonstrated easily as follows.

EX: Calculate the deflection for the cantilever beam in W.E. No.4.

SOLUTION

The deflection of the beam y is directly proportional to the force F so the work done by the force is W = Fy/2 (the aea under the F – y graph).

Equate the strain energy to the work done and $Fy/2 = 34.13$

$y = 34.13 \times 2/F$

$y = 34.13 \times 2 /800 = 0.085$ m

We can check the answer with the standard formula for the deflection of a cantilever (covered in the beams tutorials).

$$y = \frac{FL^3}{3EI} = \frac{800 \times 4^3}{3 \times 200 \times 10^3} = 0.085 \text{ m}$$

7. HARDER BEAM PROBLEMS

When the bending moment function is more complex, integrating becomes more difficult and a maths package is advisable for solving them outside of an examination. In an examination you will need to do it the hard way. For example, the bending moment function changes at every load on a simply supported beam so it should be divided up into sections and the strain energy solved for each section. The next example is typical of a solvable problem.

EX: Calculate the strain energy in the beam shown and determine the deflection under the load. The flexural stiffness is 25 MNm^2.

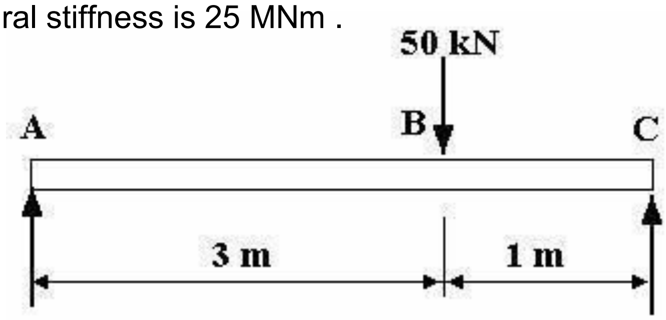


Figure 9

SOLUTION

First calculate the reactions by taking moments about the ends.

R_B x 4 = 50 x 3 R_B = 37.5 kN
R_A x 4 = 50 x 1 R_A = 12.5 kN
Check that they add up to 50 kN.

The bending moment equation is different for section AB and section BC so the solution must be done in 2 parts. The origin for x is the left end. First section AB

$M = R_A x = 12\,500\,x$

$$U = \frac{1}{2EI}\int_0^3 M^2\,dx = \frac{1}{2EI}\int_0^3 (12500x)^2\,dx$$

$$U = \frac{(12500)^2}{2 \times 25 \times 10^6}\int_0^3 x^2\,dx = \frac{(12500)^2}{2 \times 25 \times 10^6}\left[\frac{x^3}{3}\right]_0^3$$

$$U = \frac{(12500)^2}{2 \times 25 \times 10^6}\left(\frac{3^3}{3} - 0\right) = 28.125 \text{ Joules}$$

Next solve for section BC. To make this easier, let the origin for x be the right hand end.

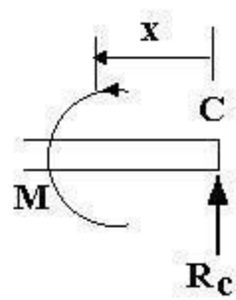


Figure 10

$M = R_B x = 37\,500\,x$

$$U = \frac{1}{2EI}\int_0^1 M^2\,dx = \frac{1}{2EI}\int_0^1 (37500x)^2\,dx$$

$$U = \frac{(37500)^2}{2 \times 25 \times 10^6}\int_0^1 x^2\,dx = \frac{(37500)^2}{2 \times 25 \times 10^6}\left[\frac{x^3}{3}\right]_0^1$$

$$U = \frac{(37500)^2}{2 \times 25 \times 10^6}\left(\frac{1^3}{3} - 0\right) = 9.375 \text{ Joules}$$

The total strain energy is U = 37.5 J
The work done by the application of the load is Fy/2 = 50 000y/2

Equating y = 0.0015 m or 1.5 mm.

EX: The diagram shows a torsion bar held rigidly at one end and with a lever arm on the other end. Solve the strain energy in the system and determine the deflection at the end of the lever arm. The force is 5000 N applied vertically. The following are the relevant stiffnesses.

Lever EI = 5 Nm^2.
Bar EI = 60 kNm^2.
Bar GJ = 50 kNm^2.

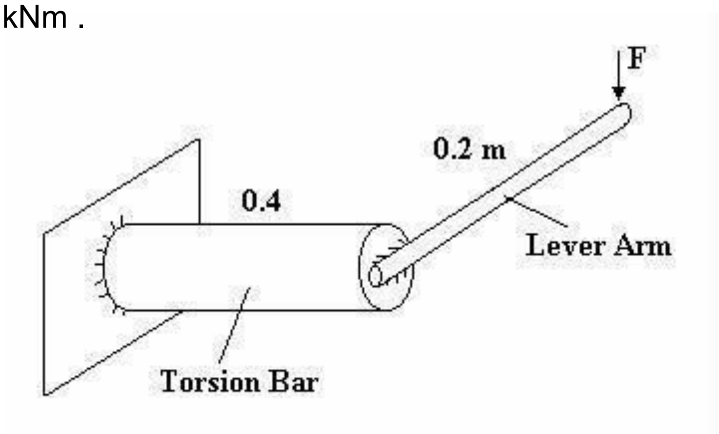


Figure 12

SOLUTION

The stresses to be considered are Bending in the lever.
Bending in the bar.
Torsion in the bar.

LEVER
Make the origin for x as shown.
The bending moment is M = Fx

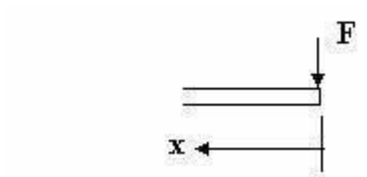


Figure 13

$$U = \frac{1}{2EI}\int_0^{0.2} M^2 dx = \frac{1}{2EI}\int_0^{0.2} (Fx)^2 dx = \frac{F^2}{2EI}\int_0^{0.2} x^2 dx = \frac{F^2}{2EI}\left[\frac{x^3}{3}\right]_0^{0.2}$$

$$U = \frac{F^2}{\quad}\left[0.2^3 - 0\right] = 266.7 \times 10^{-9} F^2 \text{ (numeric value 6.67 J)}$$

BAR

Viewed as shown we can see that the force F acts at the end of the bar as it is transmitted all along the length of the lever to the bar.

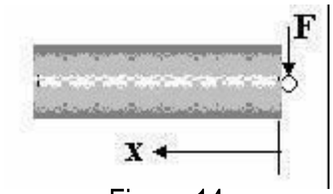


Figure 14

$$U = \frac{1}{2EI}\int_0^{0.4} M^2 dx = \frac{1}{2EI}\int_0^{0.4} (Fx)^2 dx = \frac{F^2}{2EI}\int_0^{0.4} x^2 dx = \frac{F^2}{2EI}\left[\frac{x^3}{3}\right]_0^{0.4}$$

$$U = \frac{F^2}{2 \times 60 \times 10^3 \times 3}\left[0.4^3 - 0\right] = 177.7 \times 10^{-9} F^2 \quad \text{(numeric value 4.44 J)}$$

TORSION OF BAR

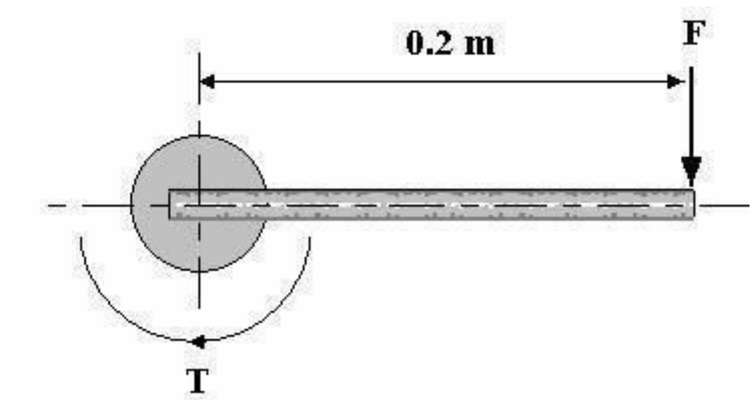


Figure 15

The torque in the bar is T = F x 0.2

For torsion

$$U = \frac{T^2 L}{2GJ} = \frac{0.04F^2}{2GJ} = \frac{0.04F^2 \times 0.4}{2 \times 50000} = 160 \times 10^{-9} F^2 \quad \text{(numeric value 4 J)}$$

The total strain energy is then $(266.7F^2 + 177.7 F^2 + 160 F^2) \times 10^{-9}$

$U = 605 \times 10^{-9} F^2$

The work done is Fy/2 so equating

$y = 2 \times 6.05 F \times 10^{-7}$

$y = 12.1 \times 5000 \times 10^{-7}$ = 0.00605 m or 6.05 mm

8. APPLICATION TO IMPACT LOADS

When a load is suddenly applied to a structure (e.g. by dropping a weight on it), the stress and deflection resulting is larger than when a static load is applied.

Consider a mass falling onto a collar at the end of a bar as shown. The bar has a length L and a cross sectional area A. The mass falls a distance z.

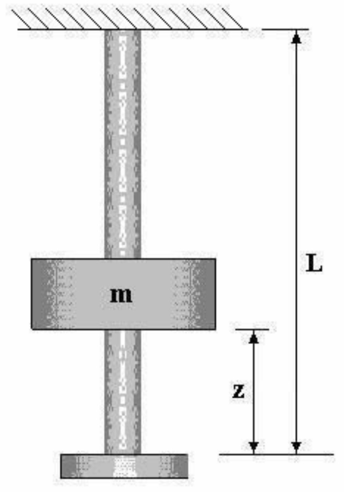


Figure 19

At the moment the bar is stretched to its maximum the force in the bar is F and the extension is x.

The corresponding stress is = F/A The strain is = x/L.

The relationship between stress and strain is E = / hence x = L/E

The strain energy in the bar is $U = {}^{2}AL/2E$

The potential energy given up by the falling mass is P.E. = mg(z + x)

8.1 SIMPLIFIED SOLUTION

If the extension x is small compared to the distance z then we may say P.E. = mgz

Equating the energy lost to the strain energy gained we have $mgz = {}^{2}AL/2E$

Hence $= \sqrt{\frac{2mgzE}{AL}}$

8.4 SUDDENLY APPLIED LOADS

A suddenly applied load occur when z =0. This is not the same as a static load. Putting z = 0 yields the result:

$$x = 2\,x_S$$

It also follows that the instantaneous stress is double the static stress.

This theory also applies to loads dropped on beams where the appropriate solution for the static deflection must be used.

EX: A mass of 5 kg is dropped from a height of 0.3 m onto a collar at the end of a bar 20 mm diameter and 1.5 m long. Determine the extension and the maximum stress induced. E = 205 GPa.

SOLUTION

$A = \pi \times 0.02^2/4 = 314.159 \times 10^{-6}\ m^2$.

$x_s = MgL/AE = 5 \times 9.81 \times 1.5 /(205 \times 10^{9} \times 314.159 \times 10^{-6}) = 1.142 \times 10^{-6}$

$x = x_s + x_s\{1 + 2z/x_s\}^{½}$

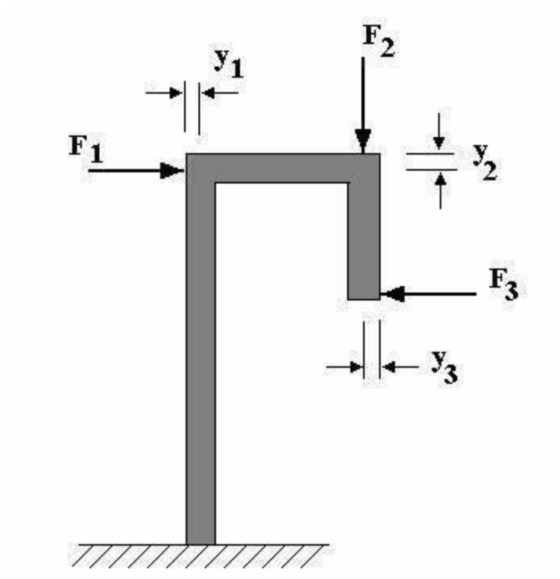
$x = 1.142 \times 10^{-6} + 1.142 \times 10^{-6}\{1 + 2 \times 0.3/1.142 \times 10^{-6}\}^{½}$

$= 828.9 \times 10^{-6}$ m

$= x\,E/L = 828.9 \times 10^{-6} \times 205 \times 10^{9}/1.5 = 113.28$ MPa

9. CASTIGLIANO'S THEOREM

Castigliano takes the work so far covered and extends it to more complex structures. This enables us to solve the deflection of structures which are subjected to several loads. Consider the structure shown.



The structure has three loads applied to it.

Consider the first point load. If the force was gradually increased from zero to F_1, the deflection would increase from zero to y_1 and the relationship would be linear as shown. The same would be true for the other two points as well.

Figure 20

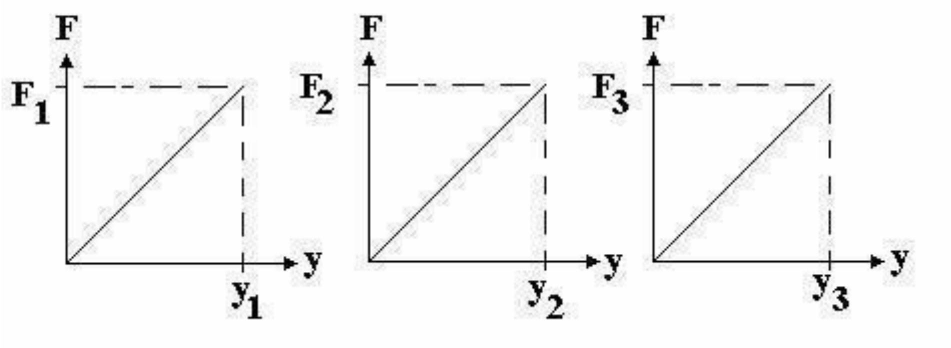


Figure 21

The work done by each load is the area under the graph. The total work is the sum of the three and this is equal to the strain energy hence:

$W = U = ½ F_1y_1 + ½ F_2y_2 + ½ F_3y_3$ .... (A) Next consider that F_1 **is further increased by** F_1 but F_2 and F_3 remain unchanged. The deflection at all three points will change and for simplicity let us suppose that they **increase as shown by** y_1, y_2 **and** y_3 respectively.

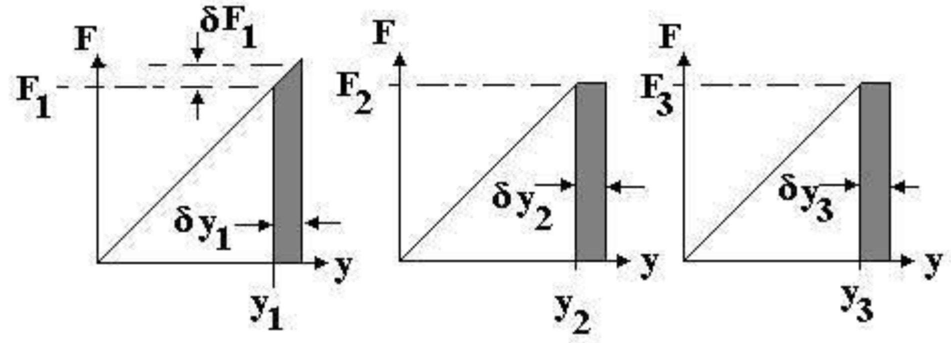


Figure 22

The increase in the work done and hence the strain energy U is represented by the shaded areas (the increase in the areas) under the graphs. Note the first one is a tall rectangle with a small triangle on top and the other two are just tall rectangles.

$$U = F_1\ y_1 + F_1\ y_1/2 + F_2\ y_2 + F_3\ y_3$$

The second term (the area of the small triangle) is very small and is ignored.

$$U = F_1\ y_1 + F_2\ y_2 + F_3\ y_3 \text{ ....(B)}$$

Now suppose that the same final points were arrived at by the gradual application of all three loads as shown.

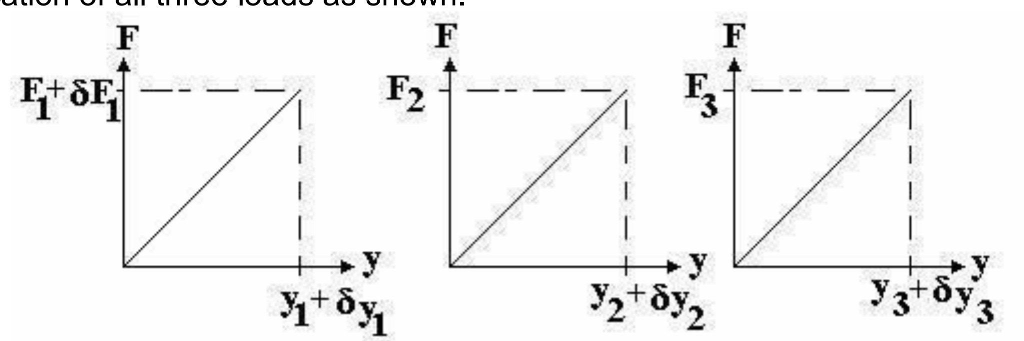


Figure 23

The work done and hence the strain energy is the area under the graphs.

$$U = ½ (F_1 + F_1)(y_1 + y_1) + ½ (F_2)(y_2 + y_2) + ½ (F_3)(y_3 + y_3) \text{ ....(C)}$$

The change in strain energy is found this time by subtracting (A) from (C). This may be equated to (B). This is a major piece of algebra that you might attempt yourself.

Neglecting small terms and simplifying we get the simple result $y_1 = U/ F_1$
Since this was found by keeping the other forces constant, we may express the equation in the form of partial differentiation since this is the definition of partial differentiation.

$$y_1 = \partial U/\partial F_1$$

If we repeated the process making F_2 change and keeping F_1 and F_3 constant we get:

$$y_2 = \partial U/\partial F_2$$

If we repeated the process making F_3 change and keeping F_1 and F_2 constant we get:

$$y_3 = \partial U/\partial F_3$$

This is Castigiano's theorem – the deflection at a point load is the partial differentiation of the rain energy with respect to that load.

Applying this is not so easy as you must determine the complete equation for the strain energy in the structure with all the forces left as unknowns until the end.

If the deflection is required at a point where there is no load, an imaginary force is placed there and then made zero at the last stage.

EX: The diagram shows a simple frame with two loads. Determine the deflection at both.

The flexural stiffness of both sections is 2 MNm^2.

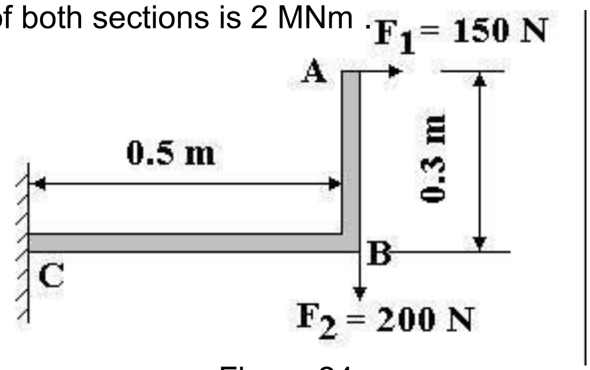


Figure 24

SOLUTION

It is important to note from the start that section AB bends and the bending moment at B turns the corner and section BC bends along its length due to both forces. Also, section BC is stretched but we will ignore this as the strain energy will be tiny compared to that produced by bending. Consider each section separately.

SECTION AB Measure the moment arm x from the free end.

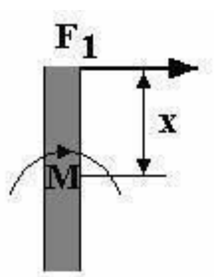


Figure 25

$M = F_1 x$ (x measured from the free end)

$$U = \frac{1}{2EI}\int_0^{0.3} M^2 dx = \frac{1}{2EI}\int_0^{0.3} (F_1 x)^2 dx = \frac{F_1^2}{2EI}\int_0^{0.3} x^2 dx$$

$$U = \frac{F_1^2}{2EI}\left[\frac{x^3}{3}\right]_0^{0.3} = \frac{F_1^2}{2 \times 2 \times 10^6}\left(\frac{0.3^3}{3} - 0\right)$$

$$U = 2.25 \times 10^{-9} F_1^2 \text{ Joules}$$

SECTION BC

The bending moment at point B is 0.3 F_1. This is carried along the section BC as a constant value. The moment am x is measured from point B. The second force produces additional bending moment of F_2 x. Both bending moments are in the same direction so they add. It is important to decide in these cases whether they add or subtract as deciding whether they are hogging (minus) or sagging (plus) is no longer relevant.

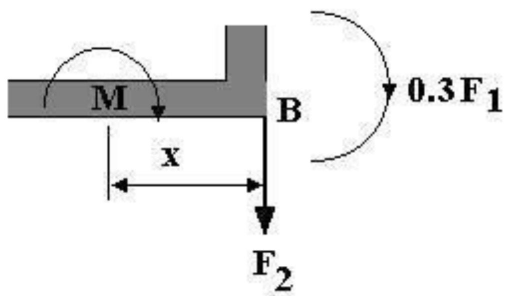


Figure 26

$M = 0.3 F_1 + F_2 x$

$$U = \frac{1}{2EI}\int_0^{0.5} M^2 dx = \frac{1}{2EI}\int_0^{0.5}(0.3F_1 + F_2 x)^2 dx = \frac{1}{2EI}\int_0^{0.5}\{(0.3F_1)^2 + (F_2^2 x^2) + (0.6F_1F_2 x)\}dx$$

$$U = \frac{1}{2EI}\left[0.09F_1^2 x + \frac{F_2^2 x^3}{3} + \frac{0.6F_1F_2x^2}{2}\right]_0^{0.5}$$

$$U = \frac{1}{2 \times 2 \times 10^6}\left\{0.09F_1^2 \times 0.5 + \frac{0.5^3 F_2^2}{3} + \frac{0.6F_1F_2 \times 0.5^2}{2}\right\}$$

$$U = 11.25F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75 F_1F_2 \times 10^{-9}$$

The total strain energy is

$$U = 11.25F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75 F_1F_2 \times 10^{-9} + 2.25 \times 10^{-9} F_1^2$$

$$U = 13.5F_1^2 \times 10^{-9} + 10.417F_2^2 \times 10^{-9} + 18.75 F_1F_2 \times 10^{-9}$$

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To find y_1 carry out partial differentiation with respect to F_1.

$$y_1 = U/ F_1 = 27F_1 \times 10^{-9} + 0 + 18.75 F_2 \times 10^{-9}$$

Insert the values of F_1 and F_2 and $y_1 = 7.8 \times 10^{-6}$ m

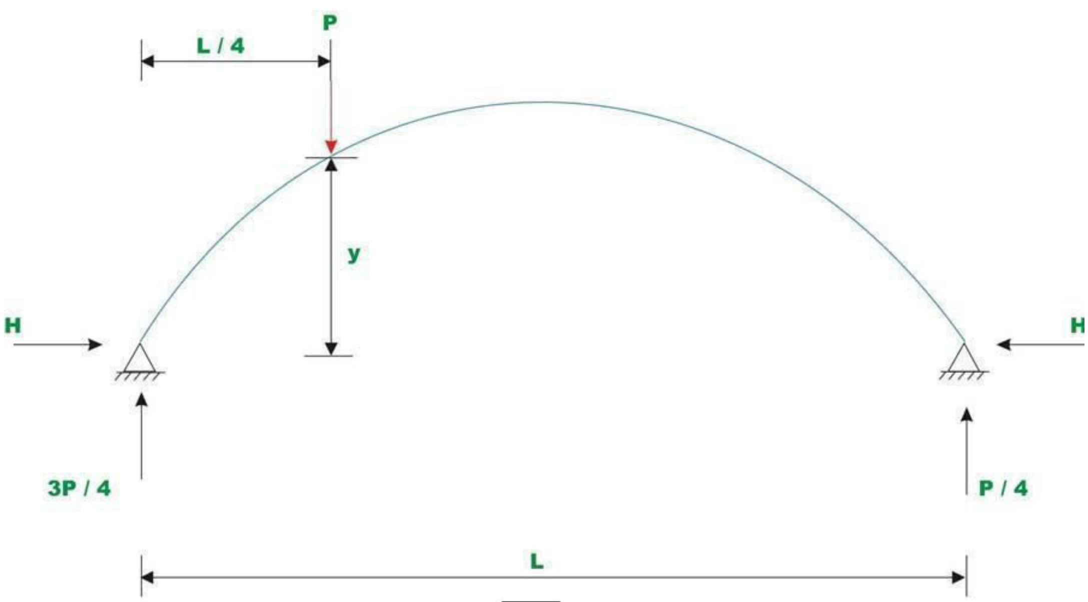
To find y_2 carry out partial differentiation with respect to F_2.

$y_2 = U/ F_2 = 0 + 20.834F_2 \times 10^{-9} + 18.75 F_1 \times 10^{-9}$ Insert the values of F_1 and F_2 and $y_2 = 7 \times 10^{-6}$ m

Three Hinged Arches

32.1 Introduction

In case of beams supporting uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.



(a) Arch

Fig. 32.1 Beam and Arch comparison.

For example, in the case 3PLofa simply supported beam shown in Fig. 32.1, the bending moment below the load is 16 . Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is3PLdetermined by the method of least work. Now the bending moment below the load is 16 *Hy* . It is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.

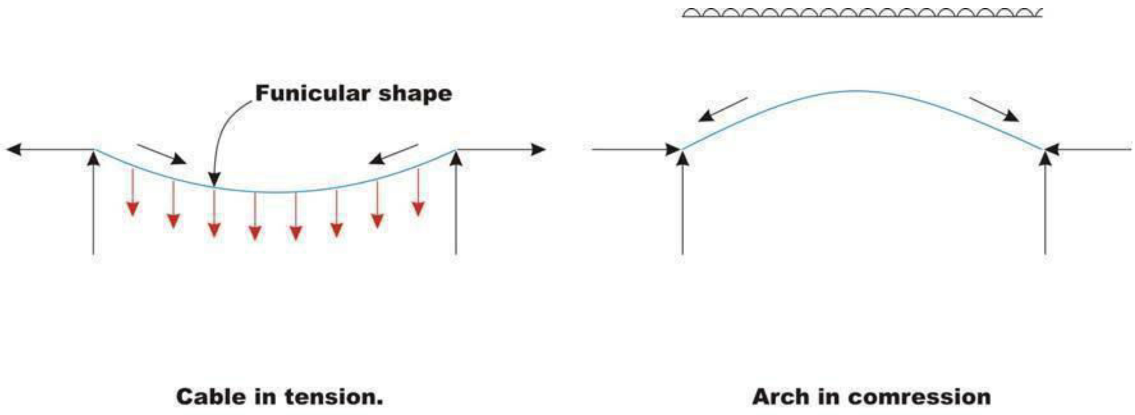


Fig. 32.2 Cable and Arch structure.

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

Until the beginning of the 20th century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and doorways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.

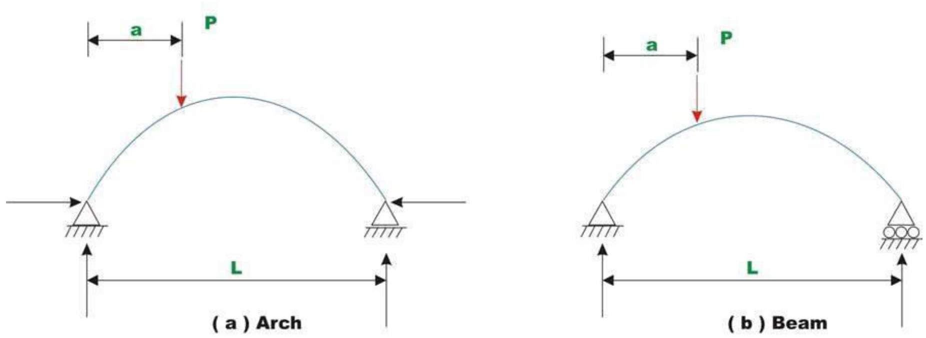


Fig. 32.3

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A structure is classified as an arch not based on its shape but the way it supports the lateral load. Arches support load primarily in compression. For example in Fig 32.3b, no horizontal reaction is developed. Consequently bending moment is not reduced. It is important to appreciate the point that the definition of an arch is a structural one, not geometrical.

32.2 Type of arches

There are mainly three types of arches that are commonly used in practice: three hinged arch, two-hinged arch and fixed-fixed arch. Three-hinged arch is statically determinate structure and its reactions / internal forces are evaluated by static equations of equilibrium. Two-hinged arch and fixed-fixed arch are statically indeterminate structures. The indeterminate reactions are determined by the method of least work or by the flexibility matrix method. In this lesson three-hinged arch is discussed.

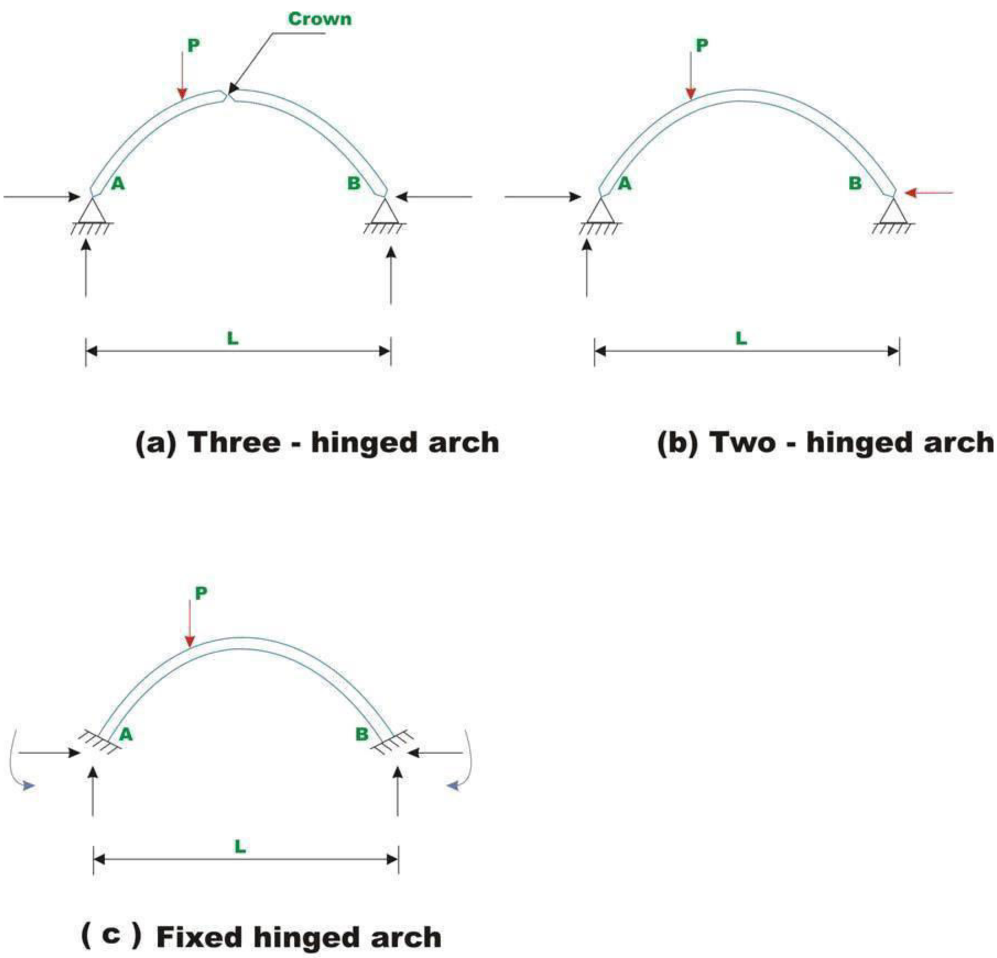

Fig. 32.4 Types of arches.

32.3 Analysis of three-hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider a three hinged arch subjected to a concentrated force *P* as shown in Fig 32.5.

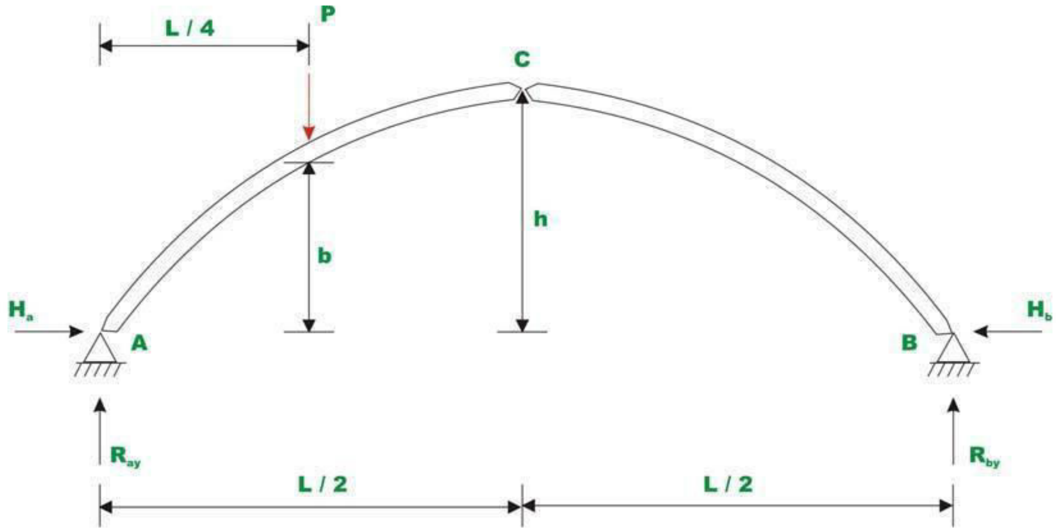

Fig. 32.5 Three hinged arch.

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium for evaluating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the required equation. Taking moment of all the forces about hinge *A* , yields

$$R_{by} = \frac{PL}{4L} = \frac{P}{4} \qquad (32.1)$$

$$\sum F_y = 0 \Rightarrow R_{ay} = \frac{3P}{4} \qquad (32.2)$$

Taking moment of all forces right of hinge *C* about hinge *C* leads to

$$H_b \times h = \frac{R_{by} L}{2}$$

$$\Rightarrow \quad \frac{R_{by} L}{} \quad \frac{PL}{}$$

$$H_b = 2h = 8h \quad (32.3)$$

Applying $\sum Fx = 0$ to the whole structure gives $H_a = \frac{PL}{8h}$

Now moment below the load is given by ,

$$M_D = \frac{R_{ay} L}{4} - H_a b$$

$$M_D = \frac{3PL}{16} - \frac{PLb}{8h} \qquad (32.4)$$

If $\frac{b}{h} = \frac{1}{2}$ then $M_D = \frac{3PL}{16} - \frac{PL}{16} = 0.125PL \qquad (32.5)$

For a simply supported beam of the same span and loading, moment under the loading is given by,

$$M_D = \frac{3PL}{16} = 0.375PL \qquad (32.6)$$

For the particular case considered here, the arch construction has reduced the moment by 66.66 %.

Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.

Solution:

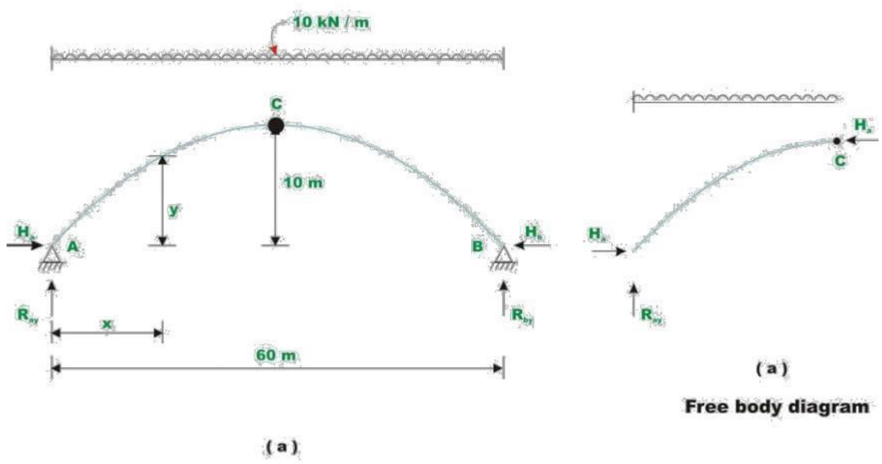

Fig. 32.6 Three hinged arch of Example 32.1

Reactions:

Taking moment of all the forces about hinge A, yields

$$R_{ay} = R_{by} = \frac{10 \times 60}{2} = 300 \text{ kN}$$

Taking moment of forces left of hinge C about C, one gets

$$R_{ay} \times 30 - H_a \times 10 - 10 \times 30 \times \frac{30}{2} = 0$$

$$H_a = \frac{300 \times 30 - 10 \times 30 \times \frac{30}{2}}{10} = 450 \text{ kN}$$

From $\sum F_x = 0$ one could write, $H_b = 450$ kN.

The shear force at the mid span is zero.

Bending moment

The bending moment at any section x from the left end is,

$$M_x = R_{ay}x - H_a y - 10\frac{x^2}{2}$$

The equation of the three-hinged parabolic arch is

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2$$

$$M_x = 300x - \left(\frac{2}{3}x - \frac{10}{30^2}x^2\right) 450 - 5x^2$$

$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression. Can you explain why the moment is zero at all points in a three-hinged parabolic arch?

Example 32.2

A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

Solution:

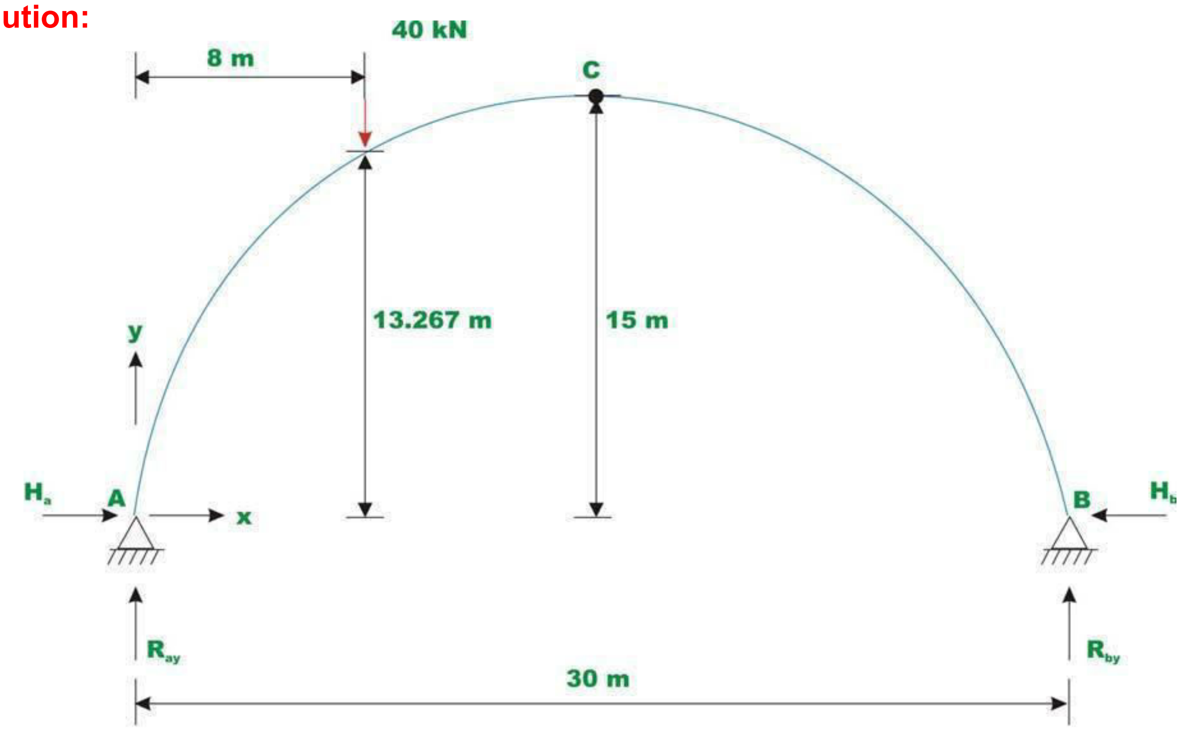

Fig. 32.7 A semi circular arch of Example 32.2

Reactions:

Taking moment of all the forces about hinge *B* leads to,

$$Ray = \frac{40 \times 22}{30} = 29.33 \text{ k}\sigma \text{ ( )}$$

$$\sum Fy = 0 \Rightarrow R_{by} = 10.67 \text{ k}\sigma \text{ ( )} \qquad (1)$$

Bending moment

Now making use of the condition that the moment at hinge *C* of all the forces left of hinge *C* is zero gives,

$$M_c = R_{ay} \times 15 - H_a \times 15 - 40 \times 7 = 0 \quad (2)$$

$$H_a = \frac{29.33 \times 15 - 40 \times 7}{15} = 10.66 \text{ kN } (\ )$$

Considering the horizontal equilibrium of the arch gives,

$H_b = 10.66$ kN ( )

The maximum positive bending moment occurs below *D* and it can be calculated by taking moment of all forces left of *D* about *D* .

$$M_D = R_{ay} \times 8 - H_a \times 13.267 \quad (3)$$

$$= 29.33 \times 8 - 10.66 \times 13.267 = 93.213 \text{ Kn}$$

Example 32.3

A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

Solution:

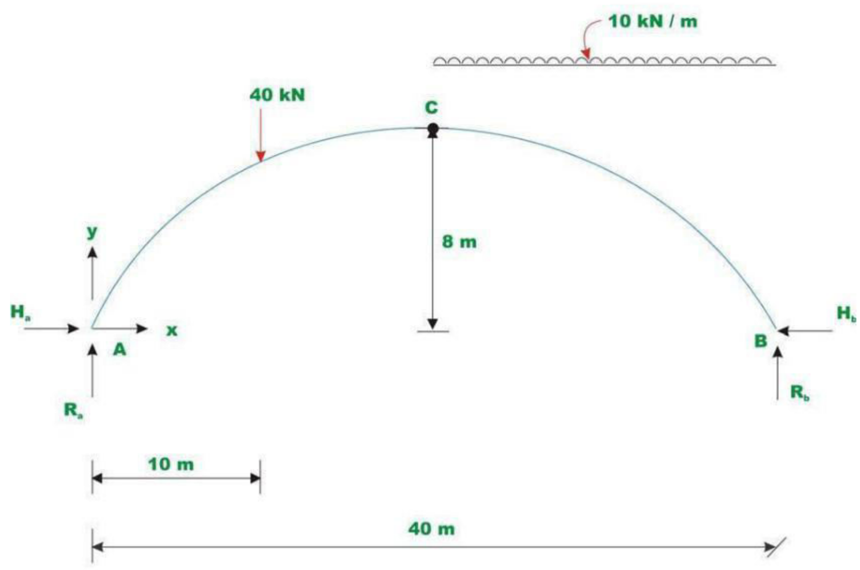

Fig. 32.8a Eaxmple 32.3

Reactions:

Taking *A* as the origin, the equation of the three-hinged parabolic arch is given by,

$$y = \frac{8}{10}x - \frac{8}{400}x^2 \quad (1)$$

Taking moment of all the forces about hinge *B* leads to,

$$R_{ay} = \frac{40\times30+10\times20\times\left(\frac{20}{2}\right)}{40} = 80 \text{ kN } (\ )$$

$$\sum F_y = 0 \Rightarrow R_{by} = 160 \text{ kN } (\ ) \quad (2)$$

Now making use of the condition that, the moment at hinge C of all the forces left of hinge C is zero gives,

$$M_C = R_{ay} \times 20 - H_a \times 8 - 40 \times 10 = 0$$

$$H_a = \frac{80 \times 20 - 40 \times 10}{8} = 150 \text{ kN ()} \qquad (3)$$

Considering the horizontal equilibrium of the arch gives,

$$H_b = 150 \text{ kN ()} \qquad (4)$$

Location of maximum bending moment

Consider a section x from end B . Moment at section x in part CB of the arch is given by (please note that B has been taken as the origin for this calculation),

$$M_x = 160x - \left(\frac{8}{10}x - \frac{8}{400}x^2\right)150 - \frac{10}{2}x^2 \qquad (5)$$

According to calculus, the necessary condition for extremum (maximum or minimum) is that $\frac{\partial M_x}{\partial x} = 0$.

$$\frac{\partial M_x}{\partial x} = 160 - \left(\frac{8}{10} - \frac{8 \times 2}{400}x\right)150 - 10x \qquad (6)$$

$$= 40 - 4x = 0$$

$$x = 10 \text{ m.}$$

Substituting the value of x in equation (5), the maximum bending moment is obtained. Thus,

$$M_{max} = 160(10) - \left(\frac{8}{10}(10) - \frac{8}{400}(10)^2\right)150 - \frac{10}{2}(10)^2$$

$$M_{max} = 200 \text{ kN.m.} \qquad (7)$$

Shear force at *D* just left of 40 kN load

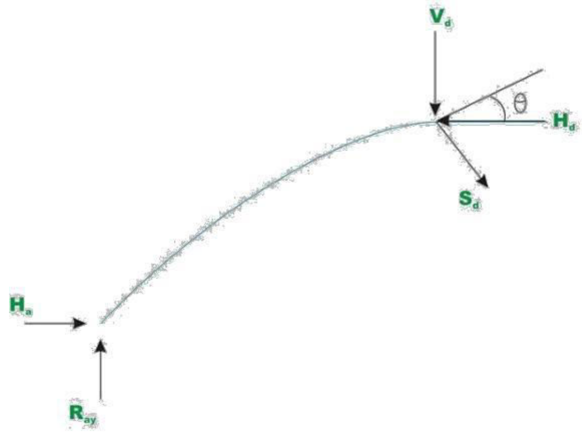

Fig. 32.8b

The slope of the arch at D is evaluated by,

$$\tan\theta = \frac{dy}{dx} = \frac{8}{10} - \frac{16}{400}x \qquad (8)$$

Substituting x =10 m. in the above equation, $\theta_D = 21.8^0$

Shear force S_d at left of D is

$$S_d = H_a \sin\theta - R_{ay}\cos\theta \qquad (9)$$

$$S_d = 150\sin(21.80) - 80\cos(21.80)$$
$$= -18.57 \text{ kN.}$$

Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch

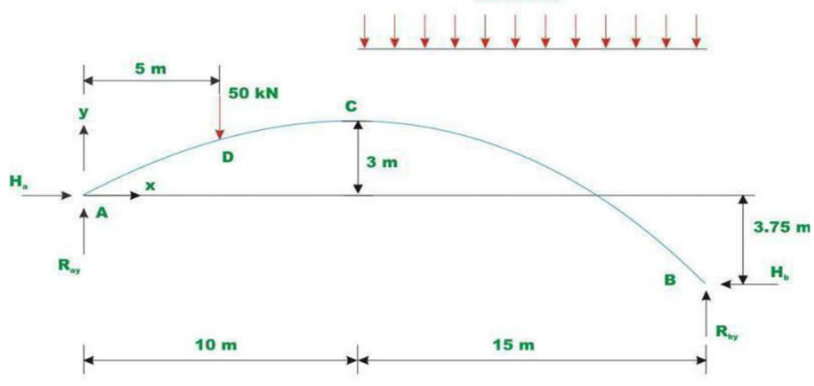

Fig. 32.9

Solution:

Reactions:

Taking A as the origin, the equation of the parabolic arch may be written as,

$$y = -0.03x^2 + 0.6x$$

Taking moment of all the loads about B leads to,

$$R_{ay} = \frac{1}{25}\left[50 \times 20 + 10 \times 15 \times \frac{15}{2} - H_a \times 3.75\right]$$

$$= \frac{1}{25}\left[2125 - 3.75 H_a\right]$$

Taking moment of all the forces right of hinge C about the hinge C and setting $M_C = 0$ leads to,

$$R_{by} \times 15 - 6.75 H_b - 10 \times 15 \times \frac{15}{2} = 0$$

$$R_{by} = \frac{1}{15}\left[1125 + 6.75 H_b\right] \qquad (3)$$

Since there are no horizontal loads acting on the arch,

$$H_a = H_b = H \ (say)$$

Applying $\sum F_y = 0$ for the whole arch,

$$R_{ay} + R_{by} = 10 \times 15 + 50 = 200$$

$$\frac{1}{25}\left[2125 - 3.75 H\right] + \frac{1}{15}\left[1125 + 6.75 H\right] = 200$$

$$85 - 0.15 H + 75 + 0.45 H = 200$$

$$H = \frac{40}{0.3} = 133.33 \text{ kN} \qquad (4)$$

From equation (2),

$$R_{ay} = 65.0 \text{ kN}$$

$$R_{by} = 135.0 \text{ kN} \qquad (5)$$

Bending moment

From inspection, the maximum negative bending moment occurs in the region *AD* and the maximum positive bending moment occurs in the region *CB* .

Span AD

Bending moment at any cross section in the span AD is

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) \qquad 0 \le x \le 5 \qquad (6)$$

For, the maximum negative bending moment in this region,

$$\frac{\partial M}{\partial x} = 0 \Rightarrow R_{ay} - H_a(-0.06x + 0.6) = 0$$

$$x = 1.8748 \text{ m}$$

$$M = -14.06 \text{ kN.m.}$$

For the maximum positive bending moment in this region occurs at *D* ,

$$M_D = R_{ay}5 - H_a(-0.03 \times 25 + 0.6 \times 5)$$

$$= +25.0 \text{ kN.m}$$

Span CB

Bending moment at any cross section, in this span is calculated by,

$$M = R_{ay}x - H_a(-0.03x^2 + 0.6x) - 50(x-5) - 10(x-10)\frac{(x-10)}{2}$$

For locating the position of maximum bending moment,

$$\frac{\partial M}{\partial x} = 0 = R_{ay} - H_a(-0.06x + 0.6) - 50 - \frac{10}{2} \times 2(x-10) = 0$$

$$x = 17.5 \text{ m}$$

$$M = 65 \times 17.5 - 133.33(-0.03(17.5)^2 + 0.6(17.5)) - 50(12.5) - \frac{10}{2}(7.5)^2$$

$$M = 56.25 \text{ kN.m}$$

Hence, the maximum positive bending moment occurs in span CB.

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Chapter 3

Propped Cantilever and Fixed Beams

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1 Introduction

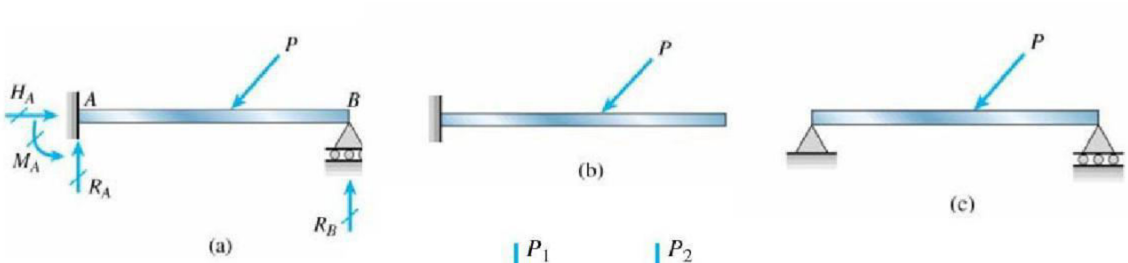
in this chapter we will analyze the beam in which the number of reactions exceed the number of independent equations of equilibrium

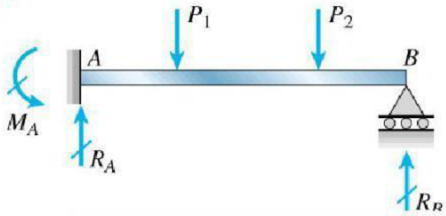
integration of the differential equation, method of superposition

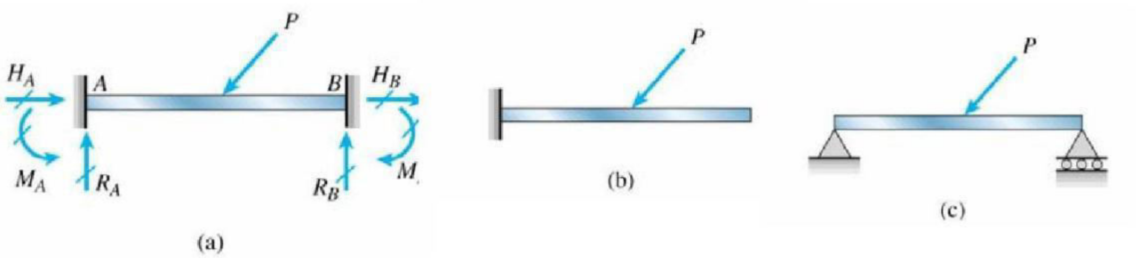
compatibility equation (consistence of deformation)

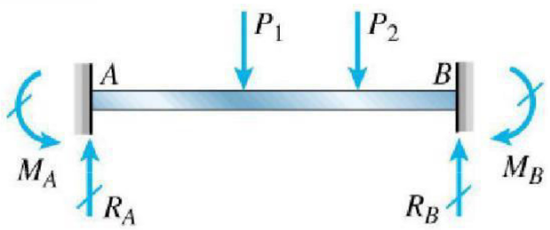
10.2 Types of Statically Indeterminate Beams

the number of reactions in excess of the number of equilibrium equations is called the degree of static indeterminacy









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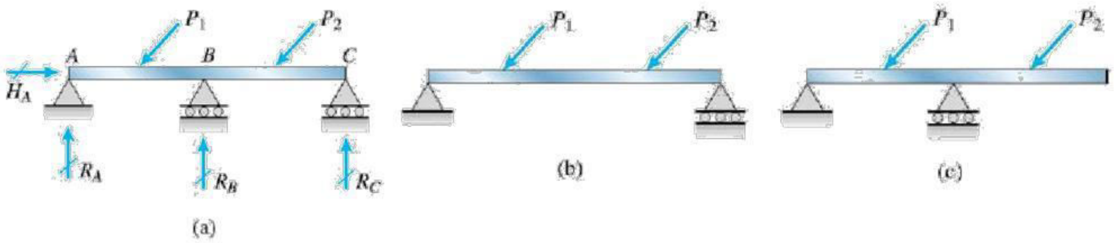


the excess reactions are called static redundants

the structure that remains when the redundants are released is called released structure or the primary structure

10.3 Analysis by the Differential Equations of the Deflection Curve

$$EIv'' = M \qquad EIv''' = V \qquad EIv^{iv} = -q$$

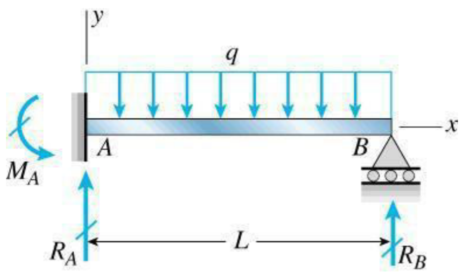
the procedure is essentially the same as that for a statically determine beam and consists of writing the differential equation, integrating to obtain its general solution, and then applying boundary and other conditions to evaluate the unknown quantities, the unknowns consist of the redundant reactions as well as the constants of integration

this method have the computational difficulties that arise when a large number of constants to be evaluated, it is practical only for relatively simple case

Example 10-1

a propped cantilever beam *AB* supports a uniform load q

determine the reactions, shear forces, bending moments, slopes, and deflections



choose R_B as the redundant, then

$R_A = qL - R_B$ $\qquad M_A = \frac{qL^2}{2} - R_B L$

and the bending moment of the beam is

$$M = R_A x - M_A - \frac{qx^2}{2}$$

$$= qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2}$$

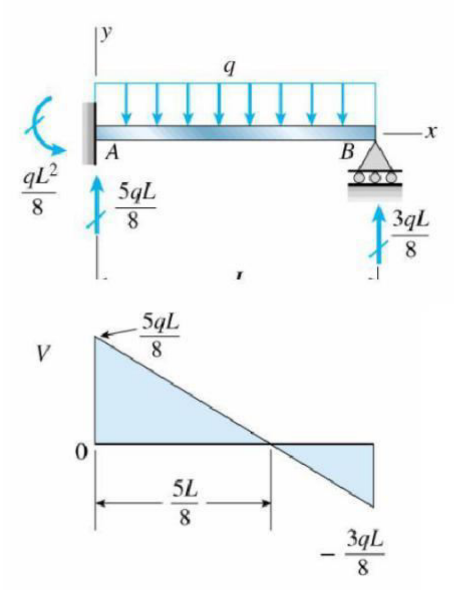
$$EIv'' = M = qLx - R_B x - \frac{qL^2}{2} - R_B L - \frac{qx^2}{2}$$

$$EIv' = \frac{qLx^2}{2} - \frac{R_B x^2}{2} - \frac{qL^2 x}{2} - R_B L x - \frac{qx^3}{6} + C_1$$

$$EIv = \frac{qLx^3}{6} - \frac{R_b x^3}{6} - \frac{qL^2 x^2}{4} - \frac{R_B L x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

boundary conditions

$v(0) = 0 \qquad v'(0) = 0 \qquad v(L) = 0$



it is obtained

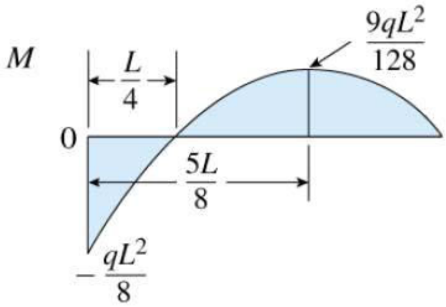
$C_1 = C_2 = 0 \qquad R_B = 3qL/8$

and $\quad R_A = 5qL/8$

$M_A = qL^2/8$

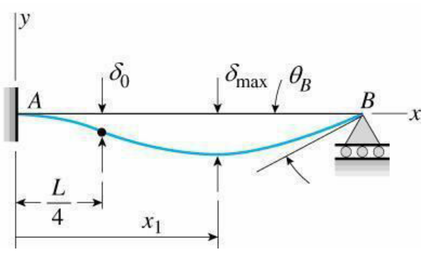
the shear force and bending moment are

$$V = R_A - qx = \frac{5qL}{8} - qx$$



$$M = R_A x - M_A - \frac{qx^2}{2}$$

$$= \frac{5qLx}{8} - \frac{qL^2}{8} - \frac{qx^2}{2}$$



the maximum shear force is

$V_{max} = 5qL/8$ at the fixed end

the maximum positive and negative moments are

$M_{pos} = 9qL^2/128$ $\quad M_{neg} = -qL^2/8$

slope and deflection of the beam

$$v' = \frac{qx}{48EI}(-6L^2 + 15Lx - 8x^2)$$

$$v = -\frac{qx^2}{48EI}(3L^2 - 5Lx + 2x^2)$$

to determine the $_{max}$, set $v' = 0$

$$-6L^2 + 15Lx - 8x^2 = 0$$

we have $x_1 = 0.5785L$

$$_{max} = -v(x_1) = 0.005416 \frac{qL^4}{EI}$$

the point of inflection is located at $M = 0$, i.e. $x = L/4$

< 0 and $M < 0$ for $x < L/4$

> 0 and $M > 0$ for $x > L/4$

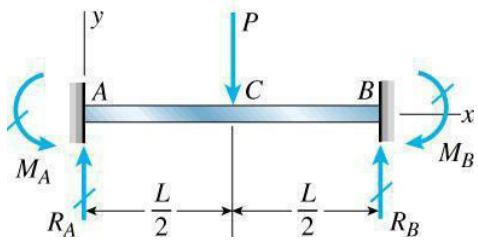
the slope at B is

$$_B = (y')_{x=L} = \frac{qL^3}{48EI}$$

Example 10-2

a fixed-end beam ABC supports a concentrated load P at the midpoint

determine the reactions, shear forces, bending moments, slopes, and deflections



because the load P in vertical direction and symmetric

$H_A = H_B = 0 \quad R_A = R_B = P/2$

$M_A = M_B$ (1 degree of indeterminacy)

$$M = \frac{Px}{2} - M_A \quad (0 \leqq x \leqq L/2)$$

$$EIv'' = M = \frac{Px}{2} - M_A \quad (0 \leqq x \leqq L/2)$$

after integration, it is obtained

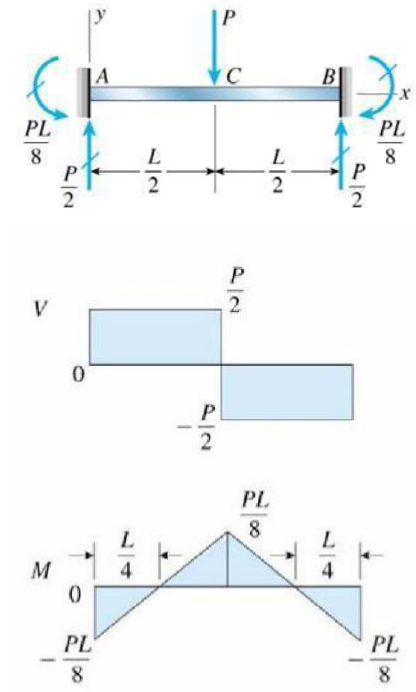
$$EIv' = \frac{Px^2}{4} - M_A x + C_1 \quad (0 \leqq x \leqq L/2)$$

$$EIv = \frac{Px^3}{12} - \frac{M_A x^2}{2} + C_1 x + C_2 \quad (0 \leqq x \leqq L/2)$$

boundary conditions

$$v(0) = 0 \qquad v'(0) = 0$$

symmetric condition



$v'(0) = 0$

the constants C_1, C_2 and the moment M_A are obtained

$$C_1 = C_2 = 0$$

$$M_A = \frac{PL}{8} = M_B$$

the shear force and bending moment diagrams can be plotted

thus the slope and deflection equations are

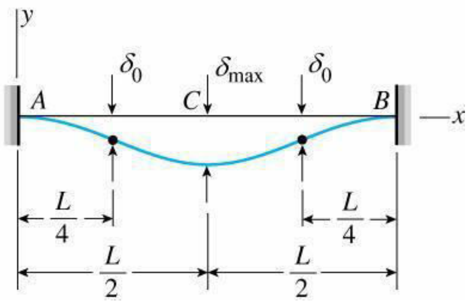
$$v' = -\frac{Px}{8EI}(L - 2x) \quad (0 \leqq x \leqq L/2)$$

$$v = -\frac{Px^2}{48EI}(3L - 4x) \quad (0 \leqq x \leqq L/2)$$

the maximum deflection occurs at the center

$$_{max} = -v(L/2) = \frac{PL^3}{192EI}$$

the point of inflection occurs at the point where $M = 0$, i.e. $x = L/4$, the deflection at this point is



$$= -v(L/4) = \frac{PL^3}{384EI}$$

which is equal $_{max}/2$

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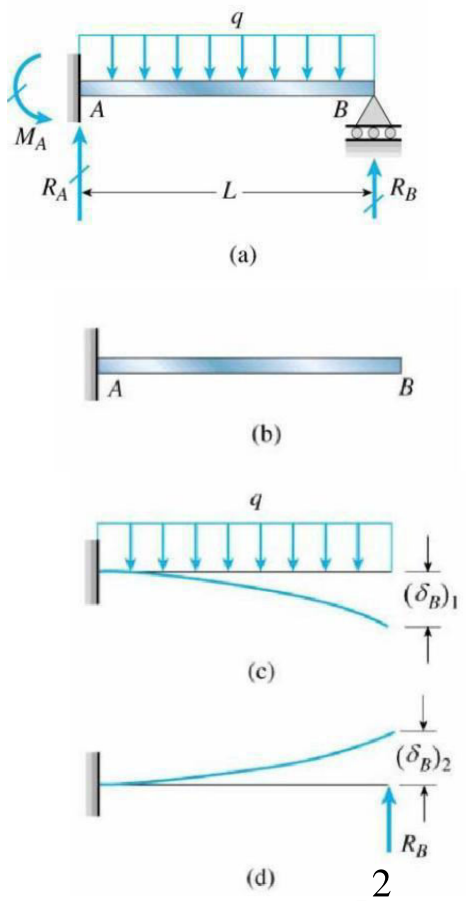
10.4 Method of Superposition

1. selecting the reaction redundants
2. establish the force-displacement relations
3. consistence of deformation (compatibility equation)



consider a propped cantilever beam

(i) select R_B as the redundant, then

$$R_A = qL - R_B \qquad M_A = \frac{qL^2}{2} - R_B L$$

force-displacement relation

$$(\delta_B)_1 = \frac{qL^4}{8EI} \qquad (\delta_B)_2 = \frac{R_B L^3}{3EI}$$

compatibility equation

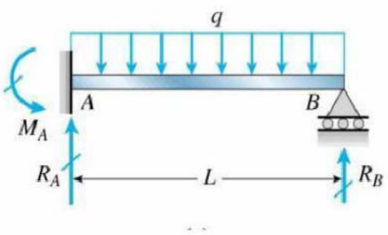
$$\delta_B = (\delta_B)_1 - (\delta_B)_1 = 0$$

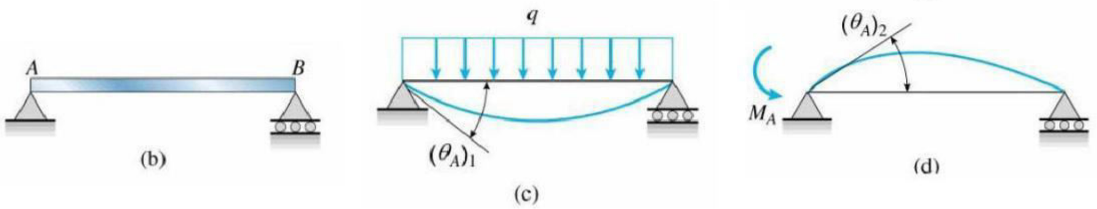
$$\frac{qL^4}{8EI} = \frac{R_B L^3}{3EI}$$

$$R_B = \frac{3qL}{8} \Rightarrow R_A = \frac{5qL}{8} \qquad M_A = \frac{qL^2}{8}$$

(ii) select the moment M_A as the redundant

$$R_A = \frac{qL}{2} + \frac{M_A}{L} \qquad R_B = \frac{qL}{2} - \frac{M_A}{L}$$





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force-displacement relation

$$(A)_1 = \frac{qL^3}{24EI} \qquad (A)_2 = \frac{M_A L}{3EI}$$

compatibility equation

$$_A = (A)_1 - (A)_2 = \frac{qL^3}{24EI} - \frac{M_A L}{3EI} = 0$$

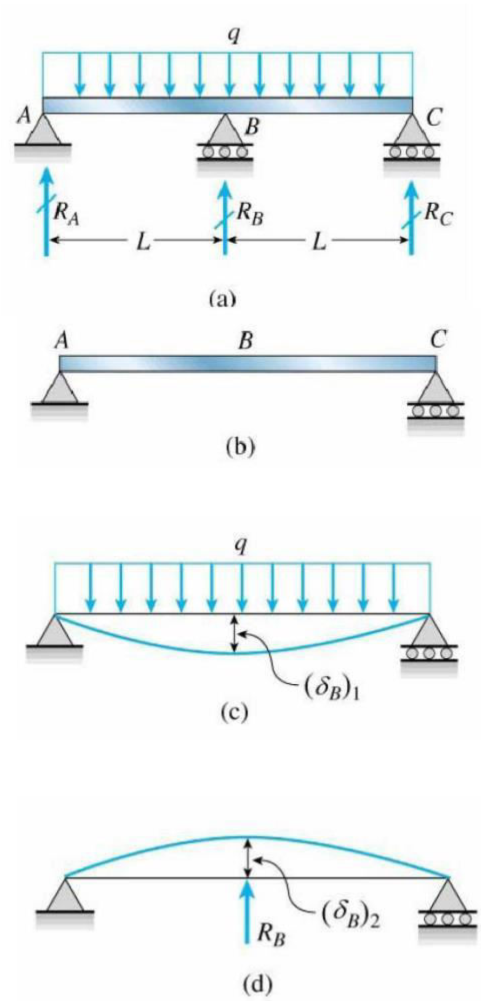
thus $M_A = qL^2/8$

and $R_A = 5qL/8$ $\qquad R_B = 3qL/8$

Example 10-3

a continuous beam ABC supports a uniform load q

determine the reactions



select R_B as the redundant, then

$$R_A = R_C = qL - \frac{qL}{2}$$

force-displacement relation

$$(\ _B)_1 = \frac{5qL(2L)^4}{384EI} = \frac{5qL^4}{24EI}$$

$$(\ _B)_2 = \frac{R_B(2L)^3}{48EI} = \frac{R_B L^3}{6EI}$$

compatibility equation

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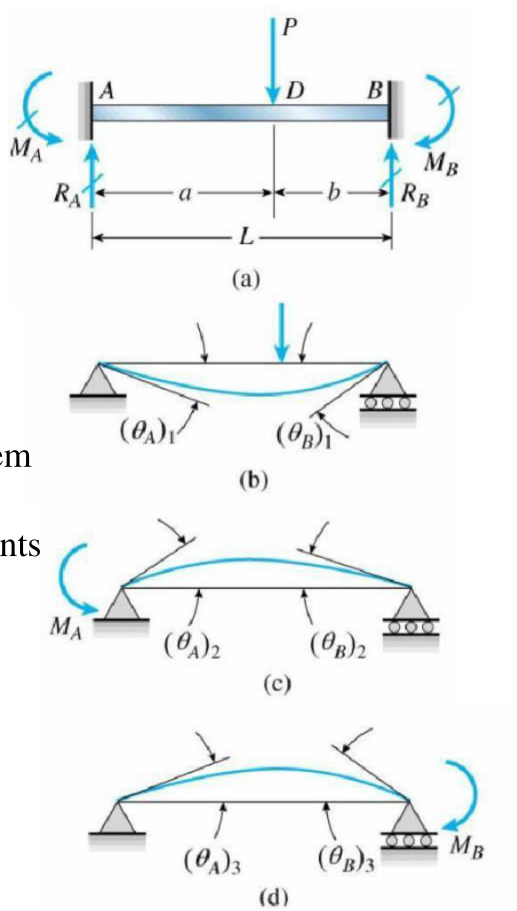
$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = \frac{5qL^4}{24EI} - \frac{R_B L^3}{6EI} = 0$$

thus $R_B = 5qL/4$

and $R_A = R_C = 3qL/8$

Example 10-4

a fixed-end beam AB is loaded by a force P acting at point D



determine reactions at the ends

also determine δ_D

this is a 2-degree of indeterminacy problem

select M_A and M_B as the redundants

$$R_A = \frac{Pb}{L} + \frac{M_A}{L} - \frac{M_B}{L}$$

$$R_B = \frac{Pa}{L} - \frac{M_A}{L} + \frac{M_B}{L}$$

force-displacement relations

$$(\theta_A)_1 = \frac{Pab(L+b)}{6LEI} \qquad (\theta_B)_1 = \frac{Pab(L+a)}{6LEI}$$

$$(\theta_A)_2 = \frac{M_A L}{3EI} \qquad (\theta_B)_2 = \frac{M_A L}{6EI}$$

$$(\theta_A)_3 = \frac{M_B L}{6EI} \qquad (\theta_B)_3 = \frac{M_B L}{3EI}$$

compatibility equations

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$$A = (A)_1 - (A)_2 - (A)_3 = 0$$

$$B = (B)_1 - (B)_2 - (B)_3 = 0$$

i.e.

$$\frac{M_A L}{3EI} + \frac{M_B L}{6EI} = \frac{Pab(L+b)}{6LEI}$$

$$\frac{M_A L}{6EI} + \frac{M_B L}{3EI} = \frac{Pab(L+a)}{6LEI}$$

solving these equations, we obtain

$$M_A = \frac{Pab^2}{L^2} \qquad M_B = \frac{Pa^2 b}{L^2}$$

and the reactions are

$$R_A = \frac{Pb^2}{L^3}(L+2a) \qquad R_B = \frac{Pa^2}{L^3}(L+2b)$$

the deflection D can be expressed as

$$D = (D)_1 - (D)_2 - (D)_3$$

$$(D)_1 = \frac{Pa^2 b^2}{3LEI}$$

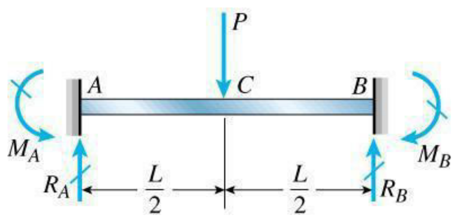
$$(D)_2 = \frac{M_A ab}{6LEI}(L + b) = \frac{Pa^2 b^3}{6L^3 EI}(L + b)$$

$$(D)_3 = \frac{M_B ab}{6LEI}(L + a) = \frac{Pa^3 b^2}{6L^3 EI}(L + a)$$

thus

$$D = \frac{Pa^3 b^3}{3L^3 EI}$$

if $a = b = L/2$



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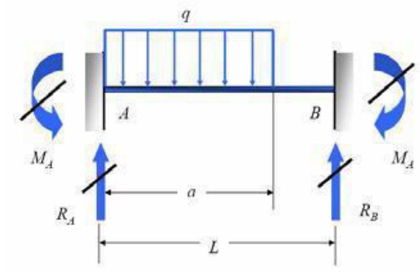
then $M_A = M_B = \dfrac{PL}{8}$ $\qquad R_A = R_B = \dfrac{P}{2}$

and $\delta_C = \dfrac{PL^3}{192EI}$

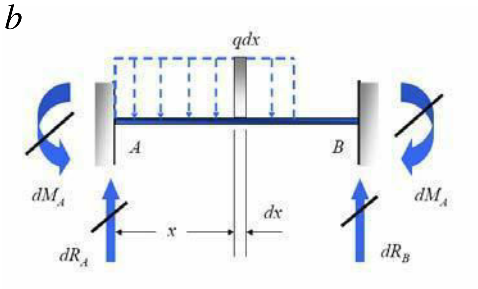
Example 10-5

a fixed-end beam AB supports a uniform load q acting over part of the span



determine the reactions of the beam

to obtain the moments caused by qdx, replace P to qdx, a to x, and b to $L - x$



$$dM_A = \frac{qx(L-x)^2 dx}{L^2}$$

$$dM_B = \frac{qx^2(L-x)dx}{L^2}$$

integrating over the loaded part

$$M_A = \int dM_A = \frac{q}{L^2}\int_0^a x(L-x)^2\,dx = \frac{qa^2}{12L^2}(6L^2 - 8aL + 3a^2)$$

$$M_B = \int dM_B = \frac{q}{L^2}\int_0^a x^2(L-x)dx = \frac{qa^3}{12L^2}(4L - 3a)$$

Similarly

$$dR_A = \frac{q(L-x)^2(L+2x)dx}{L^3}$$

$$dR_B = \frac{qx^2(3L-2x)dx}{L^3}$$

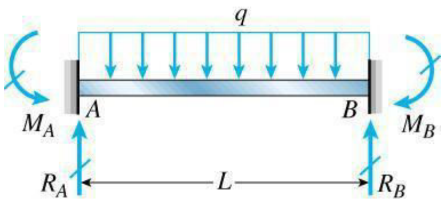
integrating over the loaded part

$$R_A = \int dR_A = \frac{q}{L^3}\int_0^a (L-x)^2(L+2x)dx = \frac{qa}{2L^3}(2L^3 - 2a^2L + a^3)$$

$$R_B = \int dR_B = \frac{q}{L^3}\int_0^a x^2(3L-2x)dx = \frac{qa^3}{2L^3}(2L-a)$$

for the uniform acting over the entire length, i.e. $a = L$

$$M_A = M_B = \frac{qL^2}{12}$$



$$R_A = R_B = \frac{qL}{2}$$

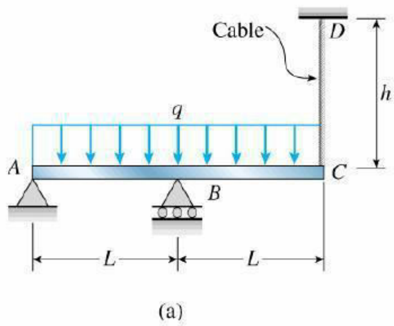
the center point deflections due to uniform load and the end moments are

$$(\delta_C)_1 = \frac{5qL^4}{384EI} \qquad (\delta_C)_2 = \frac{M_A L^2}{8EI} = \frac{(qL^2/12)L^2}{8EI} = \frac{qL^4}{96EI}$$

$$\delta_C = (\delta_C)_1 - (\delta_C)_2 = \frac{qL^4}{384EI}$$

Example 10-6

a beam ABC rests on supports A and B and is supported by a cable at C



(a)

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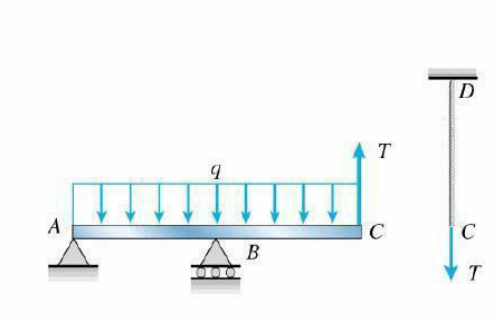
find the force T of the cable

take the cable force T as redundant the deflection $(C)_1$ due the uniform load can be found from example 9.9 with $a = L$



(b)

$$(C)_1 = \frac{qL^4}{4E_bI_b}$$

the deflection $(C)_2$ due to a force T acting on C is obtained

use conjugate beam method

$$(C)_2 = M = \frac{TL^2}{3E_bI_b} L + \frac{TL}{E_bI_b} \frac{L}{2} \frac{2L}{3}$$

$$= \frac{2TL}{3E_bI_b}$$

the elongation of the cable is

$$(C)_3 = \frac{Th}{E_cA_c}$$

compatibility equation

$$(C)_1 - (C)_2 = (C)_3$$

$$\frac{qL^4}{4E_bI_b} - \frac{2TL^3}{3E_bI_b} = \frac{Th}{E_cA_c}$$

$$T = \frac{3qL^4E_cA_c}{8L^3E_cA_c + 12hE_bI}$$

Chapter 4

Slope Deflection and Moment Distribution Method

Slope – Deflection Method

As pointed out earlier, there are two distinct methods of analysis for statically indeterminate structures depending on how equations of equilibrium, load displacement and compatibility conditions are satisfied: 1) force method of analysis and (2) displacement method of analysis. In the last module, force method of analysis was discussed. In this module, the displacement method of analysis will be discussed. In the force method of analysis, primary unknowns are forces and compatibility of displacements is written in terms of pre -selected redundant reactions and flexibility coefficients using force displacement relations. Solving these equations, the unknown redundant reactions are evaluated. The remaining reactions are obtained from equations of equilibrium.

As the name itself suggests, in the displacement method of analysis, the primary unknowns are displacements. Once the structural model is defined for the problem, the unknowns are automatically chosen unlike the force method. Hence this method is more suitable for computer implementation. In the displacement method of analysis, first equilibrium equations are satisfied. The equilibrium of forces is written by expressing the unknown joint displacements in terms of load by using load displacement relations. These equilibrium equations are solved for unknown joint displacements. In the next step, the unknown reactions are computed from compatibility equations using force displacement relations. In displacement method, three methods which are closely related to each other will be discussed.

1) Slope-Deflection Method
2) Moment Distribution Method
3) Direct Stiffness Method

In this module first two methods are discussed and direct stiffness method is treated in the next module. All displacement methods follow the above general procedure. The Slope-deflection and moment distribution methods were extensively used for many years before the compute era. After the revolution occurred in the field of computing only direct stiffness method is preferred.

Degrees of freedom

In the displacement method of analysis, primary unknowns are joint displacements which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations.These degrees of freedom are specified at supports, joints and at the free ends. For example, a propped cantilever beam (see Fig.14.01a) under the action of load P will undergo only rotation at B if axial deformation is neglected. In this case kinematic degree of freedom of the beam is only one i.e. θ_B as shown in the figure.

P_1, P_2 In Fig.14.01b, we have nodes at *A,B,C* and *D*. Under the action of lateral loads and P_3 , this continuous beam deform as shown in the figure. Here axial deformations are neglected. For this beam we have five degrees of freedom θ_A , θ_B ,θ_C , θ_D and D as indicated in the figure. In Fig.14.02a, a symmetrical plane frame is loaded symmetrically. In this case we have only two degrees of freedomθ_B andθ_C . Now consider a frame as shown in Fig.14.02b. It has three degrees of freedom viz. θ_B ,θ_C and D as shown. Under the action of horizontal and vertical load, the frame will be displaced as shown in the figure. It is observed that nodes at *B* and *C* undergo rotation and also get displaced horizontally by an equal amount.

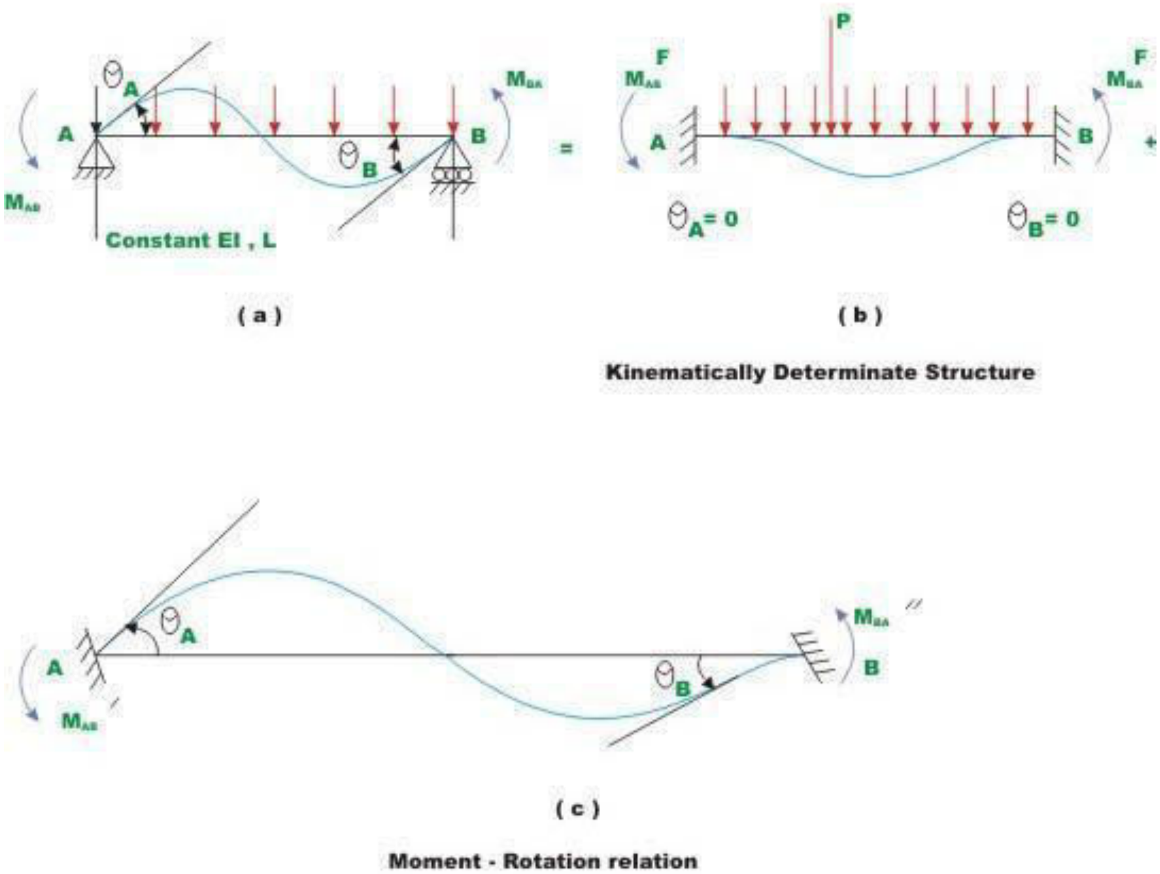


Fig.14.2 Derivation of slope - deflection equations

Hence in plane structures, each node can have at the most one linear displacement and one rotation. In this module first slope-deflection equations as applied to beams and rigid frames will be discussed.

Instructional Objectives

After reading this chapter the student will be able to
1. Calculate kinematic degrees of freedom of continuous beam.
2. Derive slope-deflection equations for the case beam with unyielding supports.
3. Differentiate between force method and displacement method of analyses.
4. State advantages of displacement method of analysis as compared to force method of analysis.
5. Analyse continuous beam using slope-deflection method.

14.1 Introduction

In this lesson the slope-deflection equations are derived for the case of a beam with unyielding supports .In this method, the unknown slopes and deflections at nodes are related to the applied loading on the structure. As introduced earlier, the slope-deflection method can be used to analyze statically determinate and indeterminate beams and frames. In this method it is assumed that all deformations are due to bending only. In other words deformations due to axial forces are neglected. As discussed earlier in the force method of analysis compatibility equations are written in terms of unknown reactions. It must be noted that all the unknown reactions appear in each of the compatibility equations making it difficult to solve resulting equations. The slope-deflection equations are not that lengthy in comparison.

The slope-deflection method was originally developed b y Heinrich Manderla and Otto Mohr for computing secondary stresses in trusses. The method as used today was presented by G.A.Maney in 1915 for analyzing rigid jointed structures.

14.2 Slope-Deflection Equations

Consider a typical span of a continuous beam AB as shown in Fig.14.1.The beam has constant flexural rigidity EI and is subjected to uniformly distributed loading and concentrated loads as shown in the figure. The beam is kinematically indeterminate to second degree. In this lesson, the slope-deflection equations are derived for the simplest case i.e. for the case of continuous beams with unyielding supports. In the next lesson, the support settlements are included in the slope-deflection equations.

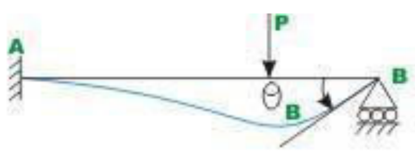


Fig. 14.01

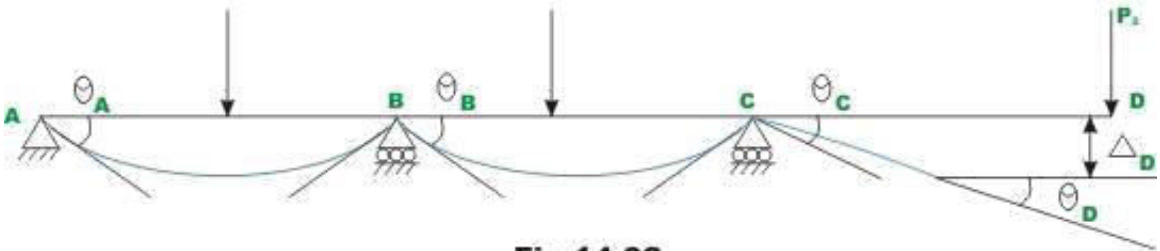


Fig.14.02

For this problem, it is required to derive relation between the joint end moments M_{AB} and M_{BA} in terms of joint rotations θ_A and θ_B and loads acting on the beam .Two subscripts are used to denote end moments. For example, end moments M_{AB} denote moment acting at joint *A* of the member *AB*. Rotations of the tangent to the elastic curve are denoted by one subscript. Thus, θ_A denotes the rotation of the tangent to the elastic curve at A. The following sign conventions are used in the slope-deflection equations (1) Moments acting at the ends of the member in counterclockwise direction are taken to be positive. (2) The rotation of the tangent to the elastic curve is taken to be positive when the tangent to the elastic curve has rotated in the counterclockwise direction from its original direction. The slope-deflection equations are derived by superimposing the end moments developed due to (1) applied loads (2) rotation θ_A (3) rotationθ_B . This is shown in Fig.14.2 (a)-(c). In Fig. 14.2(b) a kinematically determinate structure is obtained. This condition is obtained by modifying the support conditions to fixed so that the unknown joint rotations become zero. The structure shown in Fig.14.2 (b) is known as kinematically determinate structure or restrained structure. For this case, the end moments are denoted by M_{AB}^F and M_{BA}^F .

The fixed end moments are evaluated by force–method of analysis as discussed in the previous module. For example for fixed- fixed beam subjected to uniformly distributed load, the fixed-end moments are shown in Fig.14.3.

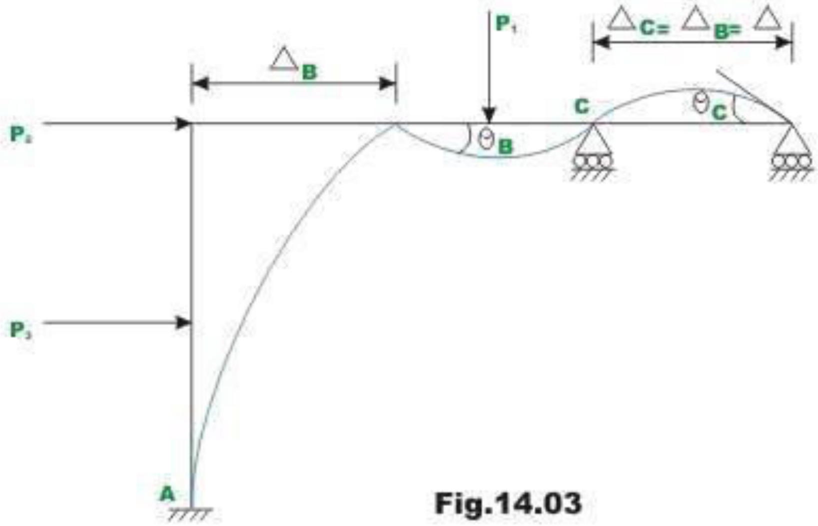


Fig.14.03

The fixed end moments are required for various load cases. For ease of calculations, fixed end forces for various load cases are given at the end of this lesson. In the actual structure end A rotates by θ_A and end B rotates by θ_B . Now it is required to derive a relation relating θ_A and θ_B with the end moments M'_{AB} and M'_{BA} . Towards this end, now consider a simply supported beam acted by moment M_{AB}' at A as shown in Fig. 14.4. The end moment M_{AB}' deflects the beam as shown in the figure. The rotations $\theta_A'$and $\theta_B'$are calculated from moment-area theorem.

$$\theta_A' = \frac{M_{AB}'L}{3EI} \qquad (14.1a)$$

$$\theta_B' = -\frac{M_{AB}'L}{6EI} \qquad (14.1b)$$

Now a similar relation may be derived if only $M_{BA}'$is acting at end B (see Fig. 14.4).

$$\theta_B'' = \frac{M_{BA}'L}{3EI} \text{ and} \qquad (14.2a)$$

$$\theta_A'' = -\frac{M_{BA}'L}{6EI} \qquad (14.2b)$$

Now combining these two relations, we could relate end moments acting at A and B to rotations produced at A and B as (see Fig. 14.2c)

$$\theta_A = \frac{M'_{AB}L}{3EI} - \frac{M'_{BA}L}{6EI} \qquad (14.3a)$$

$$\theta_B = \frac{M'_{BA}L}{3EI} - \frac{M'_{AB}L}{6EI} \qquad (14.3b)$$

Solving for M'_{AB} and M'_{BA} in terms of θ_A and θ_B,

$$M'_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.4)$$

$$M'_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.5)$$

Now writing the equilibrium equation for joint moment at A (see Fig. 14.2).

$$M_{AB} = M^F_{AB} + M'_{AB} \qquad (14.6a)$$

Similarly writing equilibrium equation for joint B

$$M_{BA} = M^F_{BA} + M'_{BA} \qquad (14.6b)$$

Substituting the value of M'_{AB} from equation (14.4) in equation (14.6a) one obtains,

$$M_{AB} = M^F_{AB} + \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.7a)$$

Similarly substituting M'_{BA} from equation (14.6b) in equation (14.6b) one obtains,

$$M_{BA} = M^F_{BA} + \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.7b)$$

Sometimes one end is referred to as near end and the other end as the far end. In that case, the above equation may be stated as the internal moment at the near end of the span is equal to the fixed end moment at the near end due to external loads plus $\frac{2EI}{L}$ times the sum of twice the slope at the near end and the slope at the far end. The above two equations (14.7a) and (14.7b) simply referred to as slope–deflection equations. The slope-deflection equation is nothing but a load displacement relationship.

14.3 Application of Slope-Deflection Equations to Statically Indeterminate Beams.

The procedure is the same whether it is applied to beams or frames. It may be summarized as follows:

1. Identify all kinematic degrees of freedom for the given problem. This can be done by drawing the deflection shape of the structure. All degrees of freedom are treated as unknowns in slope-deflection method.
2. Determine the fixed end moments at each end of the span to applied load. The table given at the end of this lesson may be used for this purpose.
3. Express all internal end moments in terms of fixed end moments and near end, and far end joint rotations by slope-deflection equations.
4. Write down one equilibrium equation for each unknown joint rotation. For example, at a support in a continuous beam, the sum of all moments corresponding to an unknown joint rotation at that support must be zero. Write down as many equilibrium equations as there are unknown joint rotations.
5. Solve the above set of equilibrium equations for joint rotations.
6. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
7. Determine all rotations.

Example 14.1

A continuous beam *ABC* is carrying uniformly distributed load of 2 kN/m in addition to a concentrated load of 20 kN as shown in Fig.14.5a. Draw bending moment and shear force diagrams. Assume *EI* to be constant.

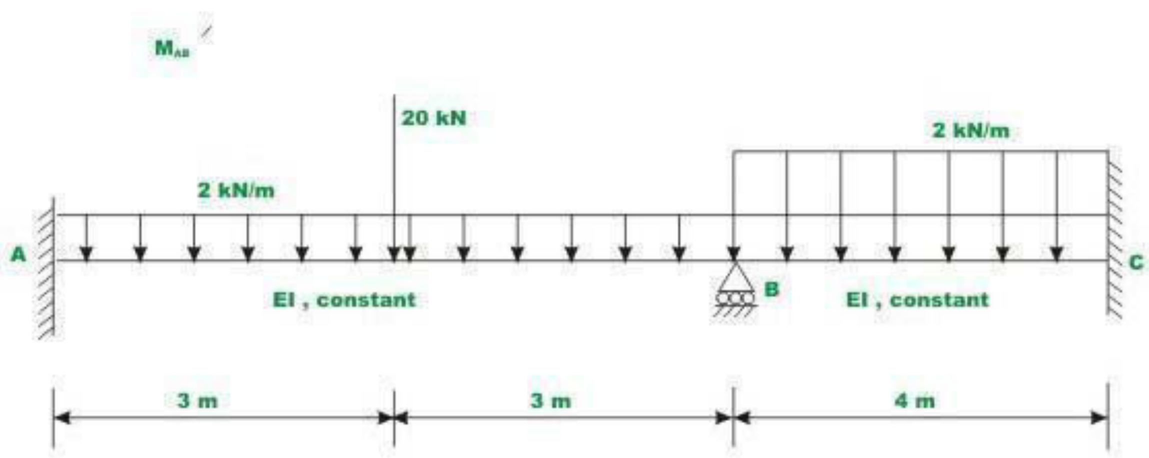


Fig. 14.5(a) Example 14.1

(a). Degrees of freedom

It is observed that the continuous beam is kinematically indeterminate to first degree as only one joint rotation θ_B is unknown. The deflected shape /elastic curve of the beam is drawn in Fig.14.5b in order to identify degrees of freedom. By fixing the support or restraining the support B against rotation, the fixed-fixed beams area obtained as shown in Fig.14.5c.

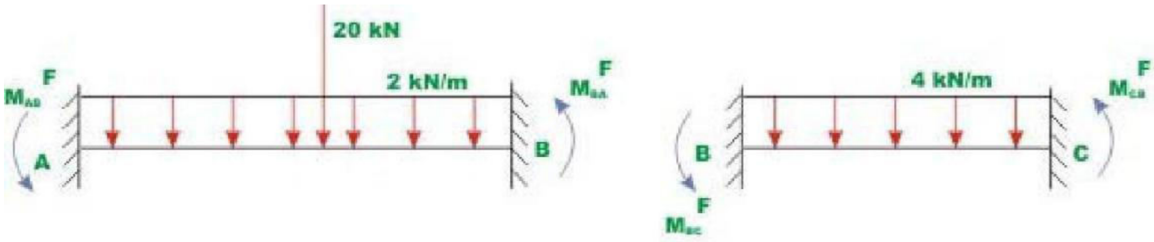


Fig. 14.5 (c) Restrained Structure.

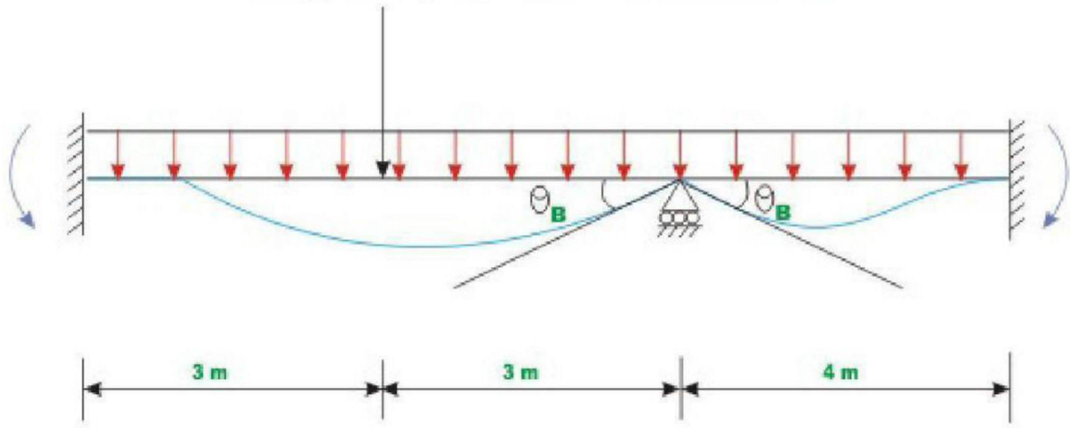


Fig. 14.5 (b) Elastic curve of the beam with unknown displacement component θ_B

(b). Fixed end moments $M_{AB}^F, M_{BA}^F, M_{BC}^F$ and M_{CB}^F are calculated referring to the Fig. 14. and following the sign conventions that counterclockwise moments are positive.

$$M_{AB}^F = \frac{2 \times 6^2}{12} + \frac{20 \times 3 \times 3^2}{6} = 21 \text{ kN . m}$$

$$M_{BA}^F = -21 \text{ kN.m}$$

$$M_{BC}^F = \frac{4 \times 4^2}{12} = 5.33 \text{ kN.m}$$

$$M_{CB}^F = -5.33 \text{ kN.m} \qquad (1)$$

(c) Slope-deflection equations

Since ends A and C are fixed, the rotation at the fixed supports is zero, $\theta_A = \theta_C = 0$. Only one non-zero rotation is to be evaluated for this problem. Now, write slope-deflection equations for span AB and BC.

$$M_{AB} = M_{AB}^F + \frac{2EI}{l}(2\theta_A + \theta_B)$$

$$M_{AB} = 21 + \frac{2EI}{6}\theta_B \qquad (2)$$

$$M_{BA} = -21 + \frac{2EI}{l}(2\theta_B + \theta_A)$$

$$M_{BA} = -21 + \frac{4EI}{6}\theta_B \qquad (3)$$

$$M_{BC} = 5.33 + EI\theta_B \qquad (4)$$

$$M_{CB} = -5.33 + 0.5EI\theta_B \qquad (5)$$

(d) Equilibrium equations

In the above four equations (2-5), the member end moments are expressed in terms of unknown rotation θ_B. Now, the required equation to solve for the rotation θ_B is the moment equilibrium equation at support B. The free body diagram of support B along with the support moments acting on it is shown in Fig. 14.5d. For, moment equilibrium at support B, one must have,

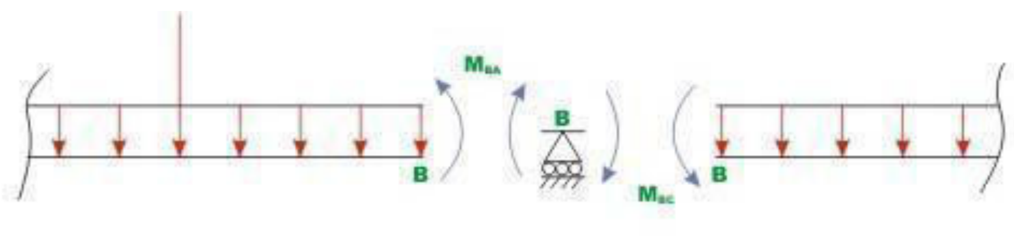


Fig. 14.5 d Free- body diagram of the joint B

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad (6)$$

Substituting the values of M_{BA} and M_{BC} in the above equilibrium equation,

$$-21 + \frac{4EI}{6}\theta_B + 5.33 + EI\theta_B = 0$$

$$\Rightarrow 1.667\theta_B EI = 15.667$$

$$\theta_B = \frac{9.398}{EI} \cong \frac{9.40}{EI} \qquad (7)$$

(e) End moments

After evaluating θ_B, substitute it in equations (2-5) to evaluate beam end moments. Thus,

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$$M_{AB} = 21 + \frac{EI}{3}\theta_B$$

$$M_{AB} = 21 + \frac{EI}{3} \times \frac{9.398}{EI} = 24.133 \text{kN.m}$$

$$M_{BA} = -21 + \frac{EI}{3}(2\theta_B)$$

$$M_{BA} = -21 + \frac{EI}{3} \times \frac{2 \times 9.4}{EI} = -14.733 \text{kN.m}$$

$$M_{BC} = 5.333 + \frac{9.4}{EI} EI = 14.733 \text{kN.m}$$

$$M_{CB} = -5.333 + \frac{9.4}{EI} \times \frac{EI}{2} = -0.63 \text{ kN.m} \qquad (8)$$

(f) Reactions

Now, reactions at supports are evaluated using equilibrium equations (vide Fig. 14.5e)

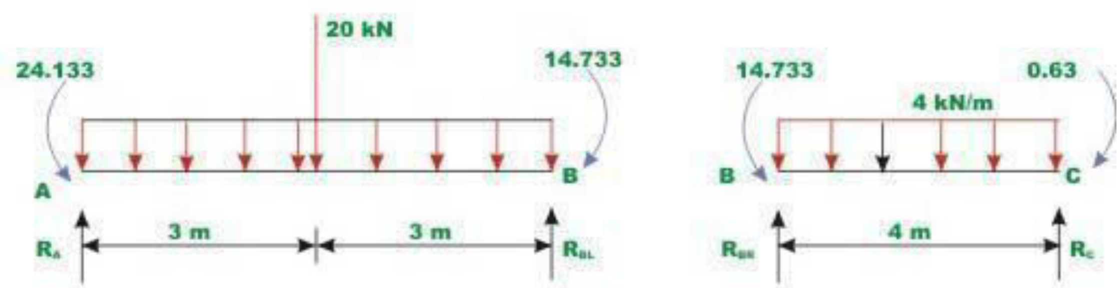

Fig. 14.5 (e) Free - body diagram of two members

$$R_A \times 6 + 14.733 - 20 \times 3 - 2 \times 6 \times 3 - 24.133 = 0$$

$$R_A = 17.567 \text{ kN}(\uparrow)$$

$$R_{BL} = 16 - 1.567 = 14.433 \text{ kN}(\uparrow)$$

$$R_{BR} = 8 + \frac{\qquad}{4} = 11.526 \text{ kN}(\uparrow)$$

$$R_C = 8 + 3.526 = 4.47 \text{ kN}(\uparrow) \qquad (9)$$

The shear force and bending moment diagrams are shown in Fig. 14.5f.

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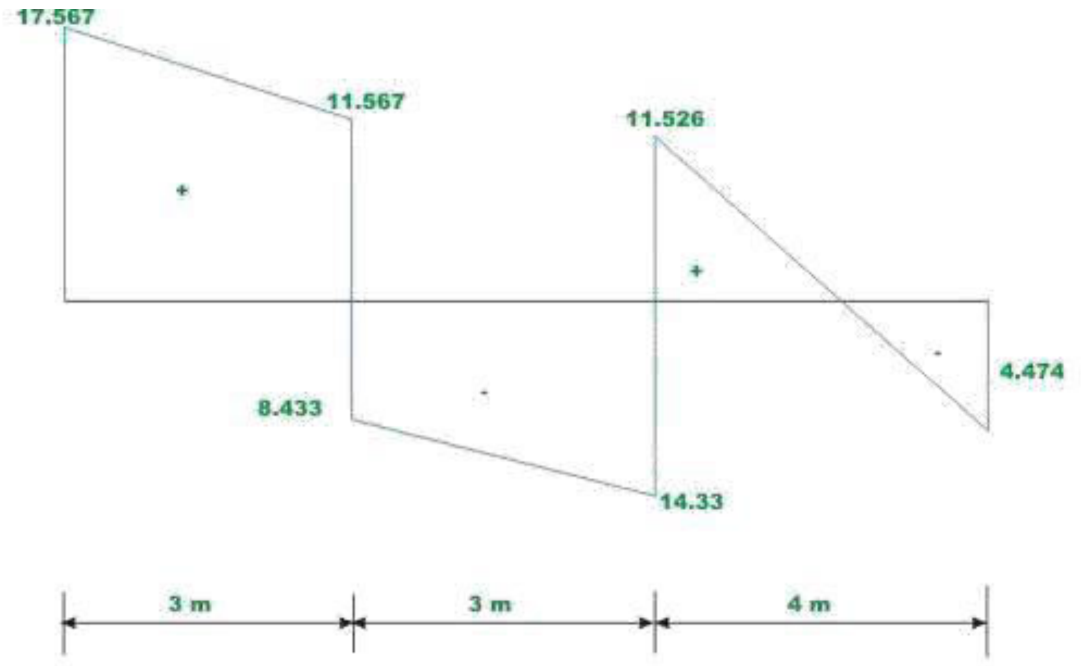
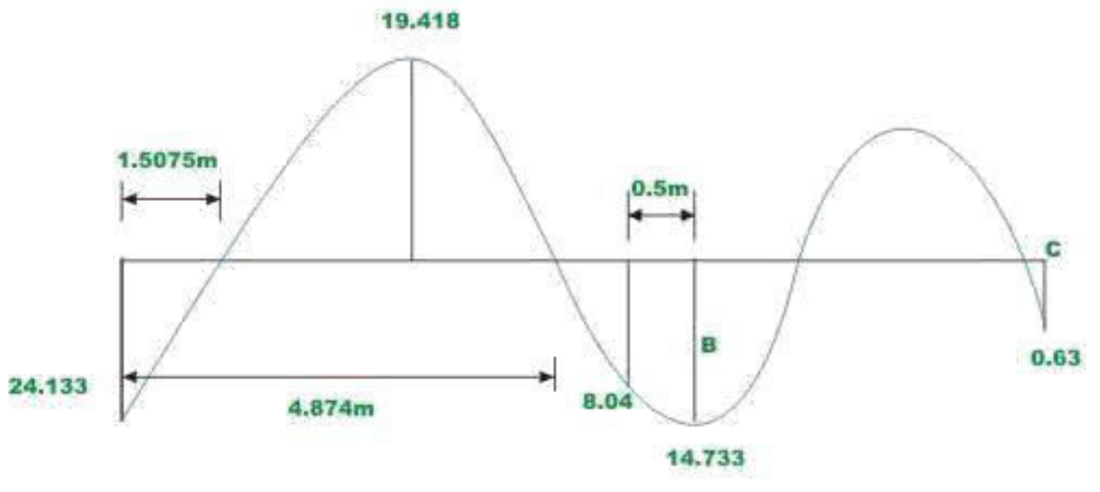


Shear force diagram



Bending Moment diagram

Fig. 14.5 f. Shear force and bending moment diagram of continuous beam ABC

Example 14.2

Draw shear force and bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.14.6a.The relative stiffness of each span of the beam is also shown in the figure.

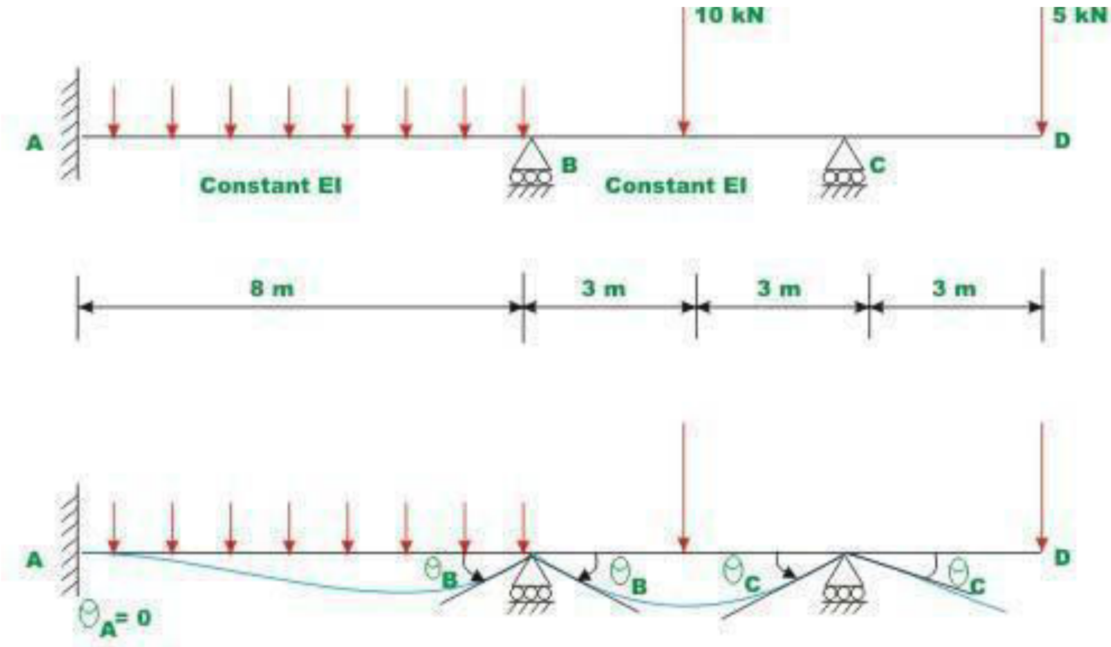


Fig. 14.6a Continuous beam of Example 14.2

For the cantilever beam portion *CD*, no slope-deflection equation need to be written as there is no internal moment at end *D*. First, fixing the supports at *B* and *C*, calculate the fixed end moments for span *AB* and *BC*. Thus,

$$M^F_{AB} = \frac{3 \times 8^2}{12} = 16 \text{ kN.m}$$

$$M^F_{BA} = -16 \text{ kN. m}$$

$$M^F_{BC} = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M^F_{CB} = -7.5 \text{ kN.m} \qquad (1)$$

In the next step write slope-deflection equation. There are two equations for each span of the continuous beam.

$$M_{AB} = 16 + \frac{2EI}{8}(\theta_B) = 16 + 0.25\theta_B EI$$

$$M_{BA} = -16 + 0.5\theta_B EI$$

$$M_{BC} = 7.5 + \frac{2 \times 2EI}{6}(2\theta_B + \theta_C) = 7.5 + 1.334EI\theta_B + 0.667EI\theta_C$$

$$M_{CB} = -7.5 + 1.334EI\theta_C + 0.667EI\theta_B \quad (2)$$

Equilibrium equations

The free body diagram of members AB , BC and joints B and C are shown in Fig.14.6b.One could write one equilibrium equation for each joint B and C.

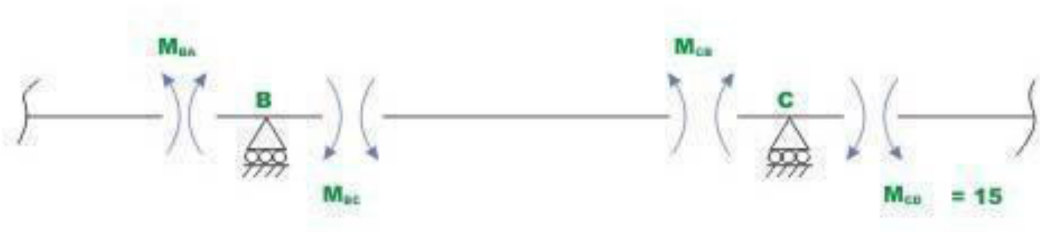


Fig. 14.6 b Free - body diagrams of joints B and C along with members

Support B,

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \quad (3)$$

$$\sum M_C = 0 \qquad M_{CB} + M_{CD} = 0 \quad (4)$$

We know that $M_{CD} = 15$ kN.m (5)

$$\Rightarrow M_{CB} = -15 \text{ kN.m} \quad (6)$$

Substituting the values of M_{CB} and M_{CD} in the above equations for M_{AB}, M_{BA}, M_{BC} and M_{CB} we get,

$$\theta_B = \frac{24.5}{3.001} = 8.164$$

$$\theta_C = 9.704 \quad (7)$$

Substituting θ_B , θ_C in the slope-deflection equations, we get

$$M_{AB} = 16 + 0.25EI\theta_B = 16 + 0.25EI \times \frac{8.164}{EI} = 18.04 \text{ kN.m}$$

$$M_{BA} = -16 + 0.5EI\theta_B = -16 + 0.5EI \times \frac{8.164}{EI} = -11.918 \text{ kN.m}$$

$$M_{BC} = 7.5 + 1.334EI \times \frac{}{EI} + 0.667EI(\frac{}{EI}) = 11.918 \text{ kN.m}$$

$$M_{CB} = -7.5 + 0.667EI \times \frac{8.164}{EI} + 1.334EI(-\frac{9.704}{EI}) = -15 \text{ kN.m} \quad (8)$$

Reactions are obtained from equilibrium equations (ref. Fig. 14.6c)

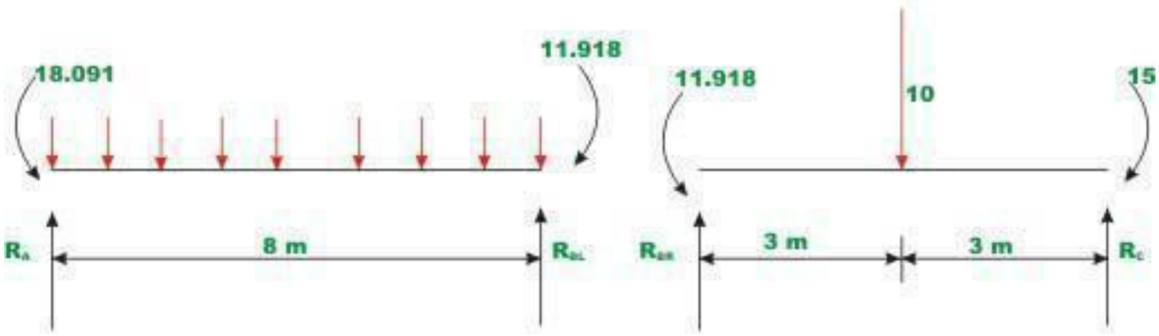


Fig. 14.6 c Computation of reactions

$$R_A \times 8 - 18.041 - 3 \times 8 \times 4 + 11.918 = 0$$

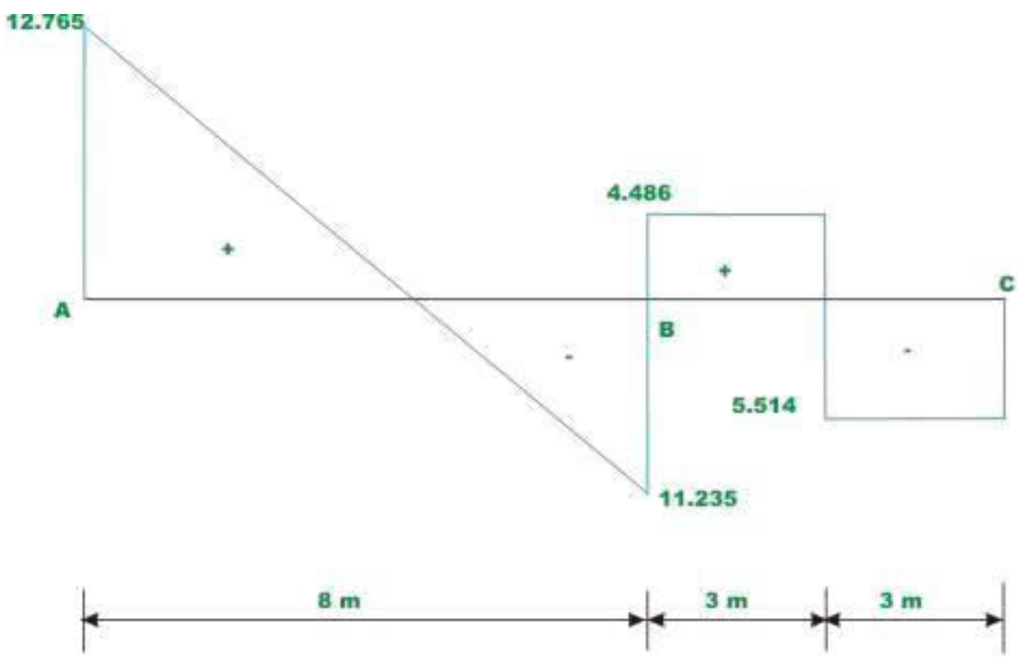
$$R_A = 12.765 \text{ kN}$$

$$R_{BR} = 5 - 0.514kN = 4.486 \text{ kN}$$

$$R_{BL} = 11.235 \text{ kN}$$

$$R_C = 5 + 0.514kN = 5.514 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 14.6d.



Shear force diagram

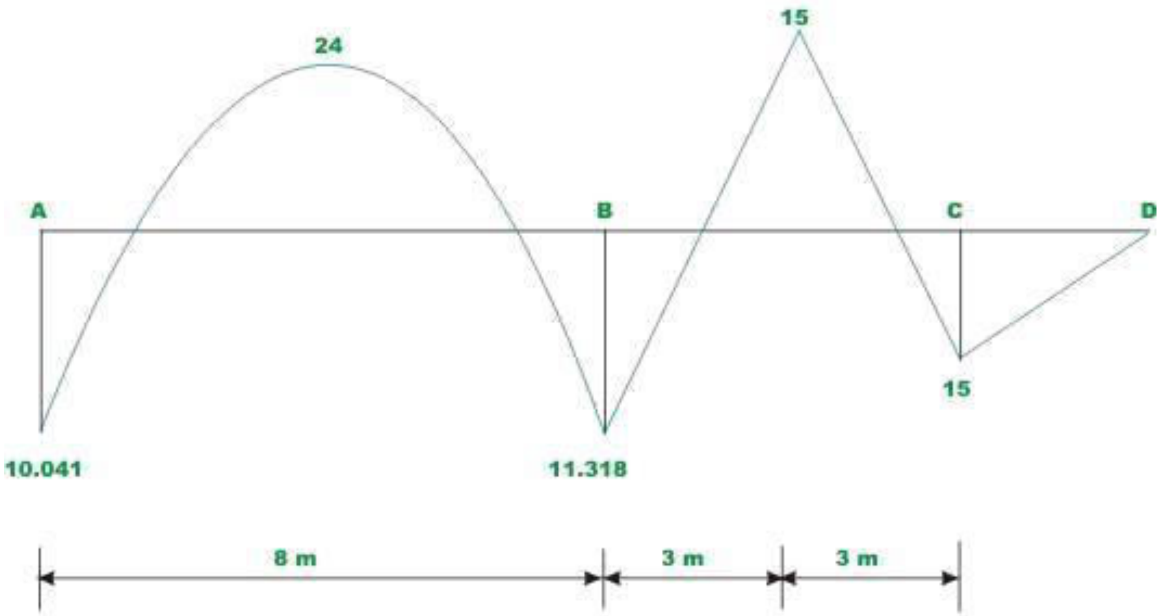


Fig. 14.6 (d) Shear force and bending moment diagram

For ease of calculations, fixed end forces for various load cases are given in Fig. 14.7.

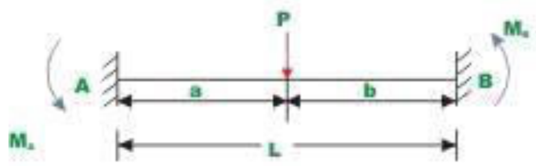
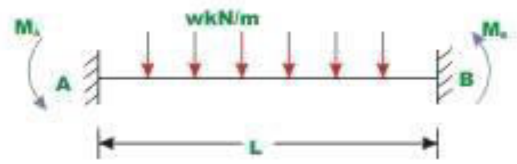
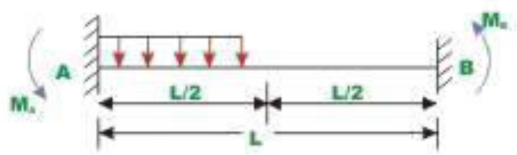
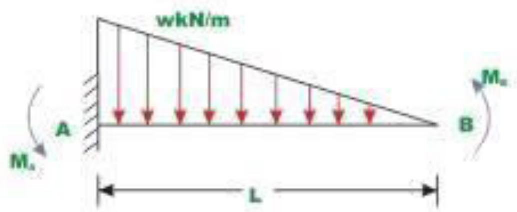
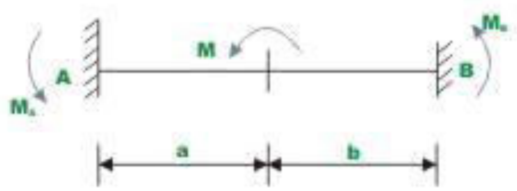
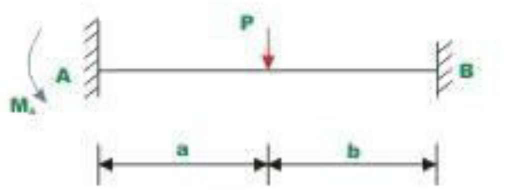
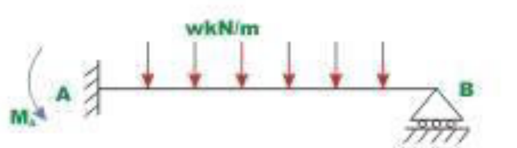
	M_A	M_B
	(+ve Counter clockwise)	
	$M_A = \frac{Pab^2}{L^2}$	$M_B = -\frac{Pab^2}{L^2}$
	$M_A = \frac{wL^2}{12}$	$M_B = -\frac{wL^2}{12}$
	$M_A = \frac{11wL^2}{192}$	$M_B = -\frac{5wL^2}{192}$
	$M_A = \frac{wL^2}{20}$	$M_B = -\frac{wL^2}{30}$
	$M_A = \frac{Mb}{L^2}(2a - b)$	$M = \frac{Ma}{L^2}(2b - a)$
	$M_A = \frac{P}{L^2}\left(b^2a + \frac{a^2b}{2}\right)$	$M_B = 0$
	$M_A = \frac{wL^2}{8}$	$M_B = 0$

Fig. 14.7 Table of fixed end moments

Summary

In this lesson the slope-deflection equations are derived for beams with unyielding supports. The kinematically indeterminate beams are analysed by slope-deflection equations. The advantages of displacement method of analysis over force method of analysis are clearly brought out here. A couple of examples are solved to illustrate the slope-deflection equations.

Module 3

Analysis of Statically Indeterminate Structures by the Displacement Method

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Lesson 18

The Moment-Distribution Method: Introduction

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Instructional Objectives

After reading this chapter the student will be able to

1. Calculate stiffness factors and distribution factors for various members in a continuous beam.
2. Define unbalanced moment at a rigid joint.
3. Compute distribution moment and carry-over moment.
4. Derive expressions for distribution moment, carry-over moments.
5. Analyse continuous beam by the moment-distribution method.

18.1 Introduction

In the previous lesson we discussed the slope-deflection method. In slope-deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope -deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method. In this lesson, first moment-distribution method is developed for continuous beams with unyielding supports.

18.2 Basic Concepts

In moment-distribution method, counterclockwise beam end moments are taken as positive. The counterclockwise beam end moments produce clockwise moments on the joint Consider a continuous beam *ABCD* as shown in Fig.18.1a.

In this beam, ends *A* and *D* are fixed and hence, $\theta_A = \theta_D = 0$ .Thus, the deformation of this beam is completely defined by rotations θ_B and θ_C at joints *B* and *C* respectively. The required equation to evaluate θ_B and θ_C is obtained by considering equilibrium of joints *B* and *C*. Hence,

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$$\sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0 \qquad (18.1a)$$

$$\sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0 \qquad (18.1b)$$

According to slope-deflection equation, the beam end moments are written as

$$M_{BA} = M_{BA}^{F} + \frac{2EI_{AB}}{L_{AB}}(2\theta_B)$$

$\frac{4EI_{AB}}{L_{AB}}$ is known as stiffness factor for the beam *AB* and it is denoted by k_{AB}. M_{BA}^{F} is the fixed end moment at joint *B* of beam *AB* when joint *B* is fixed. Thus,

$$M_{BA} = M_{BA}^{F} + K_{AB}\theta_B$$

$$M_{BC} = M_{BC}^{F} + K_{BC}\left(\theta_B + \frac{\theta_C}{2}\right)$$

$$M_{CB} = M_{CB}^{F} + K_{CB}\left(\theta_C + \frac{\theta_B}{2}\right)$$

$$M_{CD} = M_{CD}^{F} + K_{CD}\theta_C \qquad (18.2)$$

In Fig.18.1b, the counterclockwise beam-end moments M_{BA} and M_{BC} produce a clockwise moment M_B on the joint as shown in Fig.18.1b. To start with, in moment-distribution method, it is assumed that joints are locked i.e. joints are prevented from rotating. In such a case (vide Fig.18.1b),

$\theta_B = \theta_C = 0$, and hence

$$M_{BA} = M_{BA}^{F}$$

$$M_{BC} = M_{BC}^{F}$$

$$M_{CB} = M_{CB}^{F}$$

$$M_{CD} = M_{CD}^{F} \qquad (18.3)$$

Since joints *B* and *C* are artificially held locked, the resultant moment at joints *B* and *C* will not be equal to zero. This moment is denoted by M_B and is known as the unbalanced moment.

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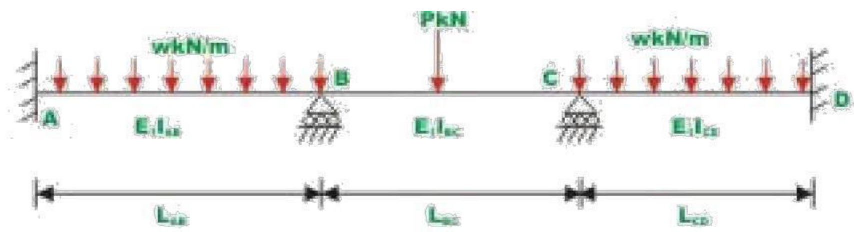

Fig. 18.1a Continuous Beam

Fig. 18.1b Continuous beam with fixed joints.

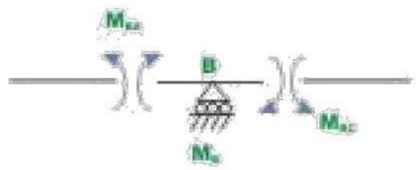

Fig. 18.1c Free - body diagram of joints B

Thus,

$$M_B = M_{BA}^F + M_{BC}^F$$

In reality joints are not locked. Joints *B* and *C* do rotate under external loads. When the joint *B* is unlocked, it will rotate under the action of unbalanced moment M_B. Let the joint B rotate by an angle θ_{B1}, under the action of M_B. This will deform the structure as shown in Fig.18.1d and introduces distributed moment M_{BA}^d, M_{BC}^d in the span BA and BC respectively as shown in the figure.

The unknown distributed moments are assumed to be positive and hence act in counterclockwise direction. The unbalanced moment is the algebraic sum of the fixed end moments and act on the joint in the clockwise direction. The unbalanced moment restores the equilibrium of the joint B. Thus,

$$\sum M_B = 0, \quad M_{BA}^d + M_{BC}^d + M_B = 0 \tag{18.4}$$

The distributed moments are related to the rotation θ_{B1} by the slope-deflection equation.

$$M_{BA}^{d} = K_{BA}\theta_{B1}$$

$$M_{BC}^{d} = K_{BC}\theta_{B1} \qquad (18.5)$$

Substituting equation (18.5) in (18.4), yields

$$\theta_{B1}(K_{BA}+K_{BC}) = -M_B$$

$$\theta_{B1} = -\frac{M_B}{K_{BA}+K_{BC}}$$

In general,

$$\theta_{B1} = -\frac{M_B}{\sum K} \qquad (18.6)$$

where summation is taken over all the members meeting at that particular joint.

Substituting the value of θ_{B1} in equation (18.5), distributed moments are calculated. Thus,

$$M_{BA}^{d} = -\frac{K_{BA}}{\sum K} M_B$$

$$M_{BC}^{d} = -\frac{K_{BC}}{\sum K} M_B \qquad (18.7)$$

The ratio $\frac{K_{BA}}{\sum K}$ is known as the distribution factor and is represented by DF_{BA}. Thus,

$$M_{BA}^{d} = -DF_{BA} \cdot M_B$$

$$M_{BC}^{d} = -DF_{BC} \cdot M_B \qquad (18.8)$$

The distribution moments developed in a member meeting at *B*, when the joint *B* is unlocked and allowed to rotate under the action of unbalanced moment M_B is equal to a distribution factor times the unbalanced moment with its sign reversed.

As the joint *B* rotates under the action of the unbalanced moment, beam end moments are developed at ends of members meeting at that joint and are known as distributed moments. As the joint *B* rotates, it bends the beam and beam end moments at the far ends (i.e. at *A* and *C*) are developed. They are known as carry over moments. Now consider the beam *BC* of continuous beam *ABCD*.

When the joint *B* is unlocked, joint *C* is locked .The joint *B* rotates by θ_{B1} under the action of unbalanced moment M_B (vide Fig. 18.1e). Now from slope-deflection equations

$$M^d_{BC} = K_{BCB}\theta$$

$$M_{CB} = \frac{1}{2} K_{BC}\theta_B$$

$$M_{CB} = \frac{1}{2} M^d_{BC} \qquad (18.9)$$

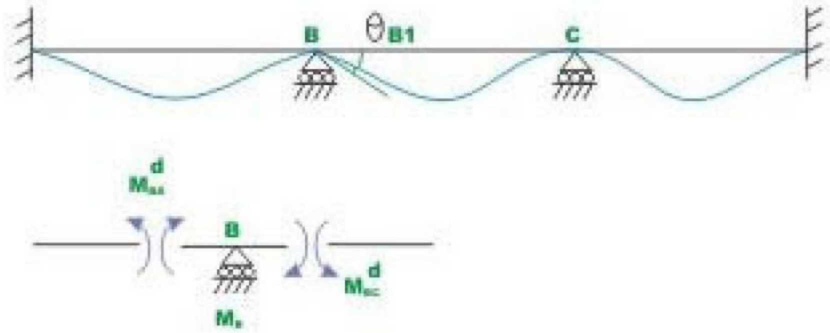

Fig. 18.1d Joint B is unlocked keeping C locked.

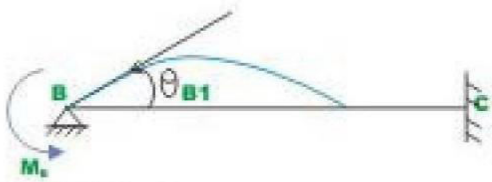

Fig.18.1e Carry - over moment

The carry over moment is one half of the distributed moment and has the same sign. With the above discussion, we are in a position to apply moment-distribution method to statically indeterminate beam. Few problems are solved here to illustrate the procedure. Carefully go through the first problem, wherein the moment-distribution method is explained in detail.

Example 18.1

A continuous prismatic beam *ABC* (see Fig.18.2a) of constant moment of inertia is carrying a uniformly distributed load of 2 kN/m in addition to a concentrated load of 10 kN. Draw bending moment diagram. Assume that supports are unyielding.

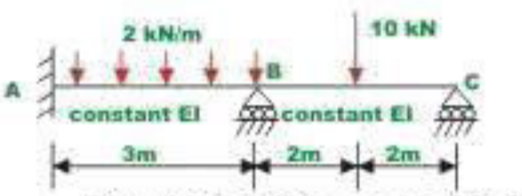

Fig. 18.2a Example 18.1

Solution

Assuming that supports *B* and C are locked, calculate fixed end moments developed in the beam due to externally applied load. Note that counterclockwise moments are taken as positive.

$$M_{AB}^F = \frac{wL_{AB}^2}{12} = \frac{2\times 9}{12} = 1.5 \text{ kN.m}$$

$$M_{BA}^F = -\frac{wL_{AB}^2}{12} = -\frac{2\times 9}{12} = -1.5 \text{ kN.m}$$

$$M_{BC}^F = \frac{Pab^2}{L_{BC}^2} = \frac{10\times 2\times 4}{16} = 5 \text{ kN.m}$$

$$M_{CB}^F = -\frac{Pa^2 b}{L_{BC}^2} = -\frac{10\times 2\times 4}{16} = -5 \text{ kN.m} \qquad (1)$$

Before we start analyzing the beam by moment-distribution method, it is required to calculate stiffness and distribution factors.

$$K_{BA} = \frac{4EI}{3}$$

$$K_{BC} = \frac{4EI}{4}$$

At *B*: $\sum K = 2.333EI$

$$DF_{BA} = \frac{1.333EI}{2.333EI} = 0.571$$

$$DF_{BC} = \frac{EI}{2.333EI} = 0.429$$

At C: $\sum K = EI$

$DF_{CB} = 1.0$

Note that distribution factor is dimensionless. The sum of distribution factor at a joint, except when it is fixed is always equal to one. The distribution moments are developed only when the joints rotate under the action of unbalanced moment. In the case of fixed joint, it does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero.

In Fig.18.2b the fixed end moments and distribution factors are shown on a working diagram. In this diagram *B* and *C* are assumed to be locked.

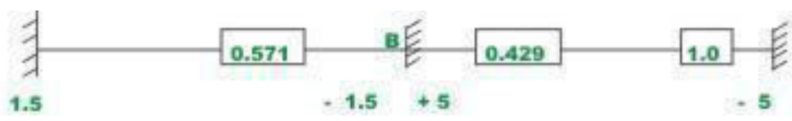

Fig. 18.2b

Now unlock the joint *C*. Note that joint *C* starts rotating under the unbalanced moment of 5 kN.m (counterclockwise) till a moment of -5 kN.m is developed (clockwise) at the joint. This in turn develops a beam end moment of +5 kN.m (M_{CB}). This is the distributed moment and thus restores equilibr ium. Now joint C is relocked and a line is drawn below +5 kN.m to indicate equilibrium. When joint *C* rotates, a carry over moment of +2.5 kN.m is developed at the B end of member *BC*.These are shown in Fig.18.2c.

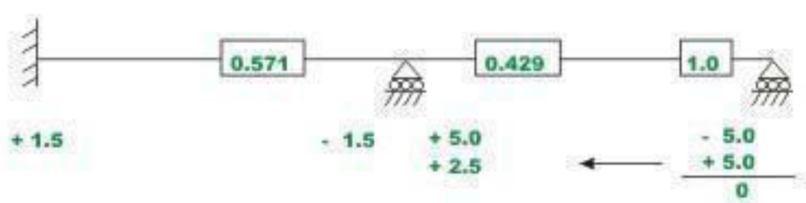

Fig. 18.2c

When joint B is unlocked, it will rotate under an unbalanced moment equal to algebraic sum of the fixed end moments(+5.0 and -1.5 kN.m) and a carry over moment of +2.5 kN.m till distributed moments are developed to restore equilibrium. The unbalanced moment is 6 kN.m. Now the distributed moments M_{BC} and M_{BA} are obtained by multiplying the unbalanced moment with the corresponding distribution factors and reversing the sign. Thus,

$M_{BC} = -2.574$ kN.m and $M_{BA} = -3.426$ kN.m. These distributed moments restore the equilibrium of joint *B*. Lock the joint *B*. This is shown in Fig.18.2d along with the carry over moments.

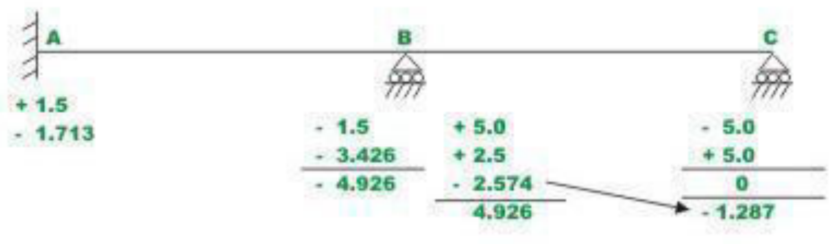

Fig. 18.2d

Now, it is seen that joint *B* is balanced. However joint *C* is not balanced due to the carry over moment -1.287 kN.m that is developed when the joint *B* is allowed to rotate. The whole procedure of locking and unlocking the joints *C* and *B* successively has to be continued till both joints B and C are balanced simultaneously. The complete procedure is shown in Fig.18.2e.

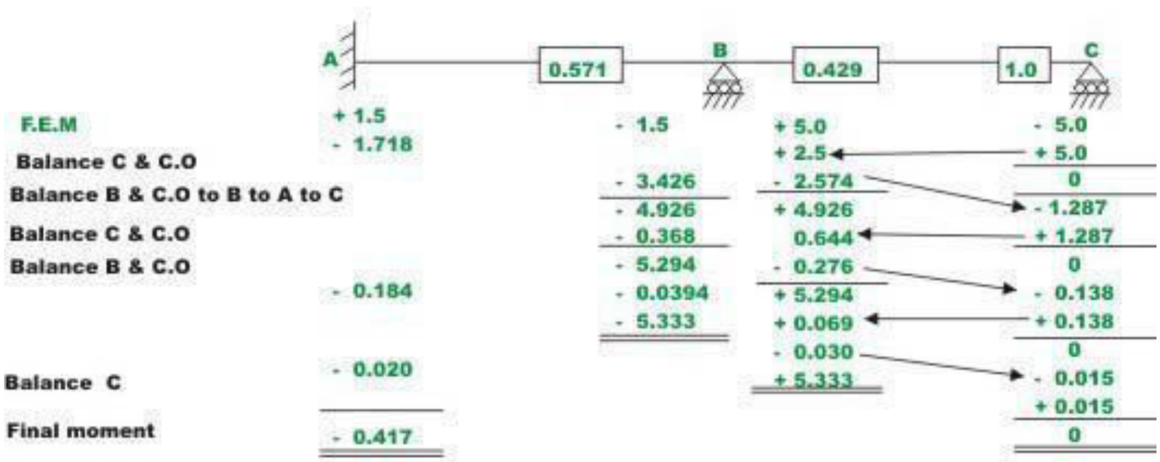

Fig. 18.2e Moment - distribution method : Computation

The iteration procedure is terminated when the change in beam end moments is less than say 1%. In the above problem the convergence may be improved if we leave the hinged end *C* unlocked after the first cycle. This will be discussed in the next section. In such a case the stiffness of beam BC gets modified. The above calculations can also be done conveniently in a tabular form as shown in Table 18.1. However the above working method is preferred in this course.

Table 18.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness	1.333EI	1.333EI	EI	EI
Distribution factor		0.571	0.429	1.0
FEM in kN.m	+1.5	-1.5	+5.0	-5.0
Balance joints C ,B and C.O.	-1.713	-3.426	+2.5 -2.579	+5.0 0
		-4.926	+4.926	-1.287
Balance C and C.O.			+0.644	1.287
Balance B and C.O.		-0.368	-0.276	-0.138
Balance C	-0.184	-5.294	+5.294	0.138
C.O.			+0.069	0
Balance B and C.O.	-0.02	-0.039	-0.030	-0.015
Balance C				+0.015
Balanced moments in kN.m	-0.417	-5.333	+5.333	0

Modified stiffness factor when the far end is hinged

As mentioned in the previous example, alternate unlocking and locking at the hinged joint slows down the convergence of moment -distribution method. At the hinged end the moment is zero and hence we could allow the hinged joint C in the previous example to rotate freely after unlocking it first time. This necessitates certain changes in the stiffness parameters. Now consider beam *ABC* as shown in Fig.18.2a. Now if joint *C* is left unlocked then the stiffness of member *BC* changes. When joint *B* is unlocked, it will rotate by θ_{B1} under the action of unbalanced moment M_B .The support *C* will also rotate by θ_{C1} as it is free to rotate. However, moment $M_{CB} = 0$. Thus

$$M_{CB} = K_{BC}\,\theta_C + \frac{K_{BC}}{2}\theta_B \tag{18.7}$$

But, $M_{CB} = 0$

$$\Rightarrow \theta_C = -\frac{\theta_B}{2} \tag{18.8}$$

Now,

$$M_{BC} \quad K_{BC} \; B \quad \frac{K_{BC}}{2}\theta_C \tag{18.9}$$

Substituting the value of θ_C in eqn. (18.9),

$$M_{BC} = \frac{3 K_{BC}}{4}\theta_B = \frac{3}{4} K_{BC}\theta_B \qquad (18.10)$$

$$M_{BC} = K^R_{BC}\theta_B \qquad (18.11)$$

The K^R_{BC} is known as the reduced stiffness factor and is equal to $\frac{3}{4}K_{BC}$. Accordingly distribution factors also get modified. It must be noted that there is no carry over to joint *C* as it was left unlocked.

Example 18.2

Solve the previous example by making the necessary modification for hinged end *C*.

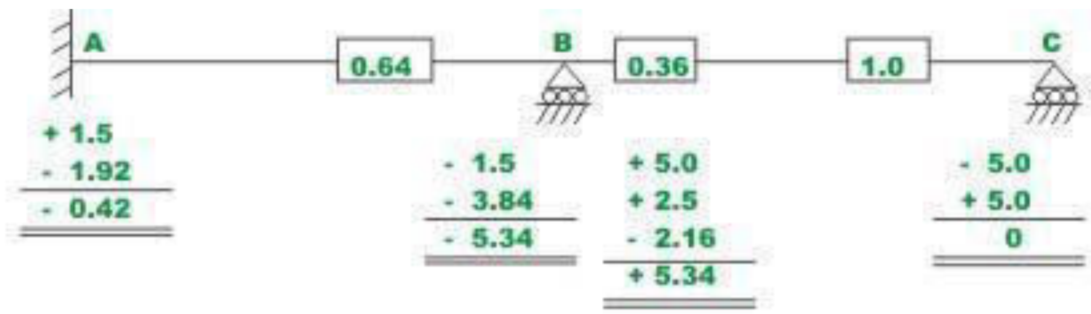

Fig. 18.3 Example 18.2

Fixed end moments are the same. Now calculate stiffness and distribution factors.

$$K_{BA} = 1.333EI,\ K^R_{BC} = \frac{3}{4}EI = 0.75EI$$

Joint B: $\sum K = 2.083$, $D^{BA}_F = 0.64$, $D^{BC}_F = 0.36$

Joint C: $\sum K = 0.75EI$, $D^{CB}_F = 1.0$

All the calculations are shown in Fig.18.3a

Please note that the same results as obtained in the previous example are obtained here in only one cycle. All joints are in equilibrium when they are unlocked. Hence we could stop moment-distribution iteration, as there is no unbalanced moment anywhere.

Example 18.3

Draw the bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.18.4a.The relative moment of inertia of each span of the beam is also shown in the figure.

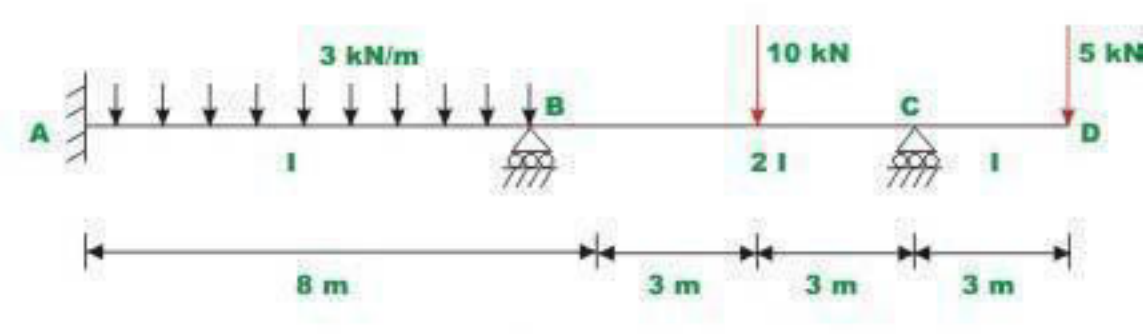

Fig. 18.4a Example 18.3

Solution

Note that joint *C* is hinged and hence stiffness factor *BC* gets modified. Assuming that the supports are locked, calculate fixed end moments. They are

$$M_{AB}^{F} = 16 \text{ kN.m}$$

$$M_{BA}^{F} = -16 \text{ kN.m}$$

$$M_{BC}^{F} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m} \text{ , and}$$

$$M_{CD}^{F} = 15 \text{ kN.m}$$

In the next step calculate stiffness and distribution factors

$$K_{BA} = \frac{4EI}{8}$$

$$K_{BC} = \frac{3\ 8EI}{4\ \ 6}$$

$_{CB}$ 6

At joint B:

$$\sum K = 0.5EI + 1.0EI = 1.5EI$$

$$D^F_{BA} = \frac{0.5\,EI}{1.5\,EI} = 0.333$$

$$D^F_{BC} = \frac{1.0\,EI}{1.5\,EI} = 0.667$$

At *C*:

$$\sum K = EI, \quad D^F_{CB} = 1.0$$

Now all the calculations are shown in Fig.18.4b

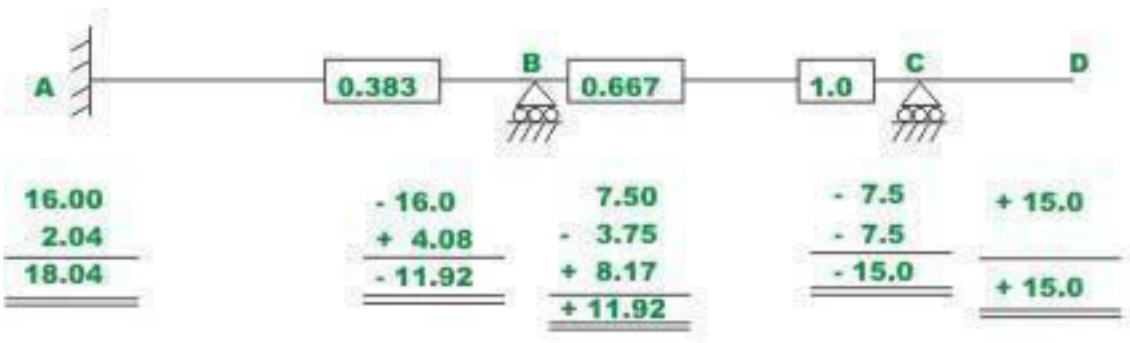

Fig. 18.4b Computation

This problem has also been solved by slope-deflection method (see example 14.2).The bending moment diagram is shown in Fig.18.4c.

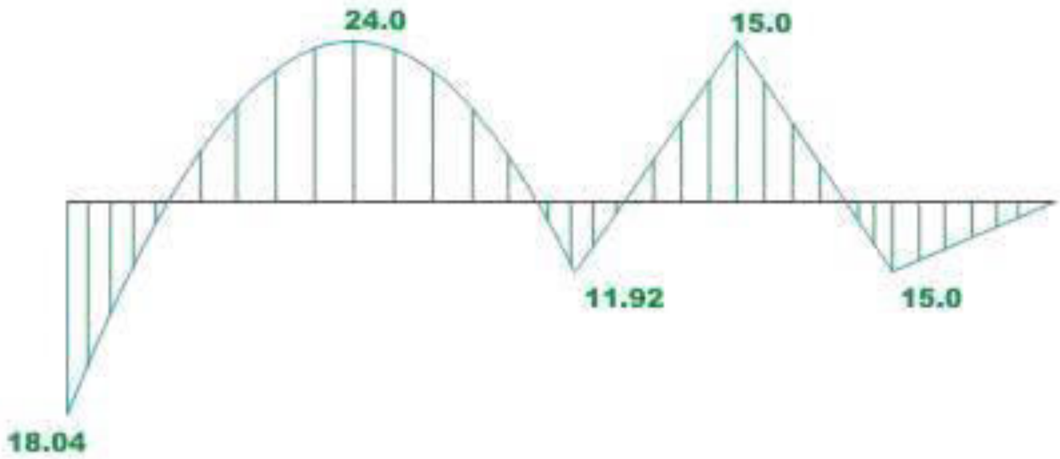

Fig. 18.4c Bending - moment diagram

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Chapter 5

Moving Loads and Influence Lines

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Instructional Objectives:

The objectives of this lesson are as follows:

- Understand the moving load effect in simpler term
- Study various definitions of influence line
- Introduce to simple procedures for construction of influence lines

37.1 Introduction

In earlier lessons, you were introduced to statically determinate and statically indeterminate structural analysis under non-moving load (dead load or fixed loads). In this lecture, you will be introduced to determination of maximum internal actions at cross-sections of members of statically determinate structured under the effects of moving loads (live loads).

Common sense tells us that when a load moves over a structure, the deflected shape of the structural will vary. In the process, we can arrive at simple conclusion that due to moving load position on the structure, reactions value at the support also will vary.

From the designer's point of view, it is essential to have safe structure, which doesn't exceed the limits of deformations and also the limits of load carrying capacity of the structure.

37.2 Definitions of influence line

In the literature, researchers have defined influence line in many ways. Some of the definitions of influence line are given below.

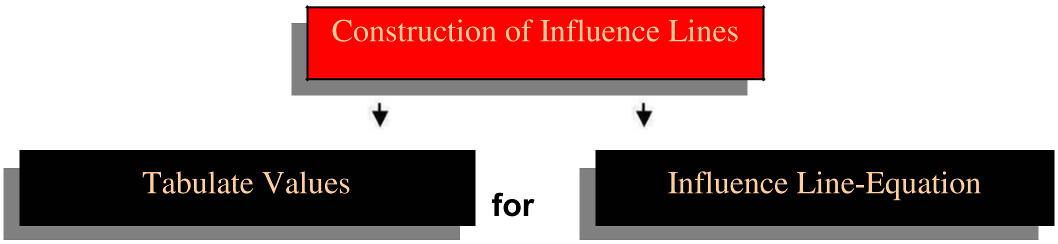
- An influence line is a diagram whose ordinates, which are plotted as a function of distance along the span, give the value of an internal force, a reaction, or a displacement at a particular point in a structure as a unit load move across the structure.
- An influence line is a curve the ordinate to which at any point equals the value of some particular function due to unit load acting at that point.
- An influence line represents the variation of either the reaction, shear,

moment, or deflection at a specific point in a member as a unit concentrated force moves over the member.

37.3 Construction of Influence Lines

In this section, we will discuss about the construction of influence lines. Using any one of the two approaches (Figure 37.1), one can construct the influence line at a specific point P in a member for any parameter (Reaction, Shear or

Moment). In the present approaches it is assumed that the moving load is having dimensionless magnitude of unity. Classification of the approaches for construction of influence lines is given in Figure 37.1.

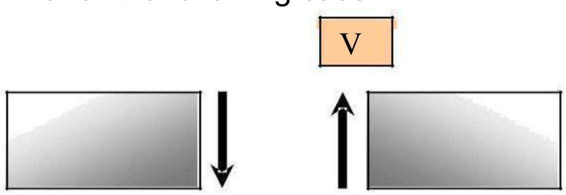
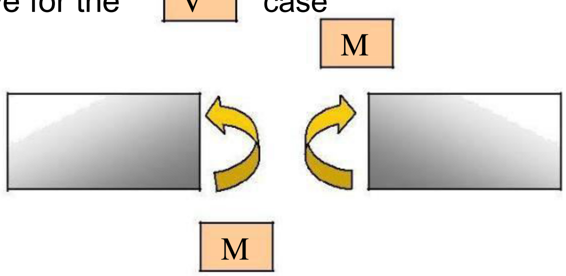


37.3.1 Tabulate Values

Apply a unit load at different locations along the member, say at ***x***. And these locations, apply statics to compute the value of parameter (reaction, shear, or moment) at the specified point. The best way to use this approach is to prepare a table, listing unit load at x versus the corresponding value of the parameter calculated at the specific point (i.e. Reaction R, Shear V or moment M) and plot the tabulated values so that influence line segments can be constructed.

37.3.2 Sign Conventions

Sign convention followed for shear and moment is given below.

Parameter	Sign for influence line
Reaction R	Positive at the point when it acts upward on the beam.
Shear V	Positive for the following case 
Moment M	Positive for the V case 

37.3.3 Influence Line Equations

Influence line can be constructed by deriving a general mathematical equation to compute parameters (e.g. reaction, shear or moment) at a specific point under the effect of moving load at a variable position ***x***.

The above discussed both approaches are demonstrated with the help of simple numerical examples in the following paragraphs.

37.4 Numerical Examples

Example 1:

Construct the influence line for the reaction at support B for the beam of span 10 m. The beam structure is shown in Figure 37.2.

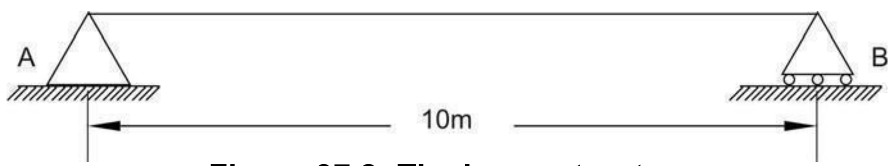


Figure 37.2: The beam structure

Solution:

As discussed earlier, there are two ways this problem can be solved. Both the approaches will be demonstrated here.

Tabulate values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction RB can be calculated as follows (Figure 37.3).

$$\Sigma M_A = 0 : R_B \times 10 - 1 \times 2.5 = 0 \Rightarrow R_B = 0.25$$

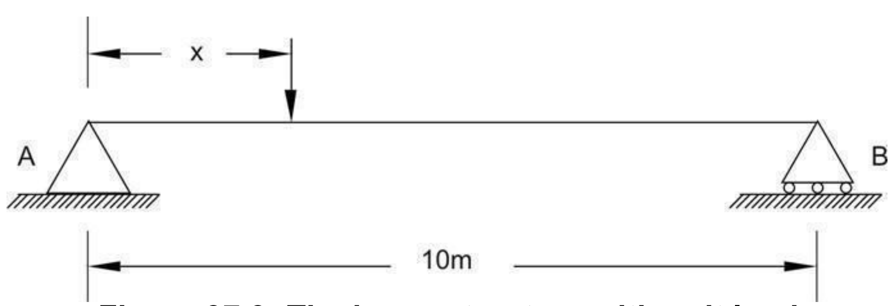


Figure 37.3: The beam structure with unit load

Similarly, the load can be placed at 5.0, 7.5 and 10 m. away from support A and reaction R_B can be computed and tabulated as given below.

x	R_B
0	0.0
2.5	0.25
5.0	0.5
7.5	0.75
10	1

Graphical representation of influence line for R_B is shown in Figure 37.4.

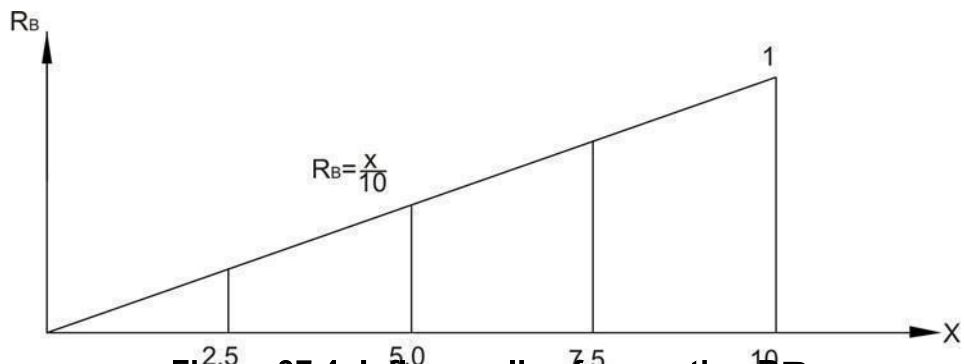


Figure 37.4: Influence line for reaction R_B.

Influence Line Equation:

When the unit load is placed at any location between two supports from support A at distance x then the equation for reaction R_B can be written as

$$\Sigma M_A = 0 : R_B \times 10 - x = 0 \Rightarrow R_B = x/10$$

The influence line using this equation is shown in Figure 37.4.

Example 2:

Construct the influence line for support reaction at B for the given beam as shown in Fig 37.5.

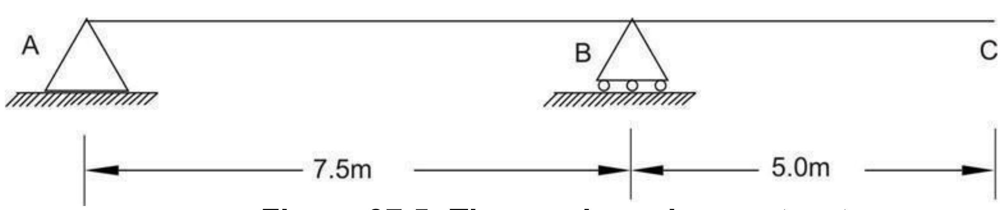


Figure 37.5: The overhang beam structure

Solution:

As explained earlier in example 1, here we will use tabulated values and influence line equation approach.

Tabulate Values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction R_B can be calculated as follows.

$$\Sigma M_A = 0 : R_B \times 7.5 - 1 \times 2.5 = 0 \Rightarrow R_B = 0.33$$

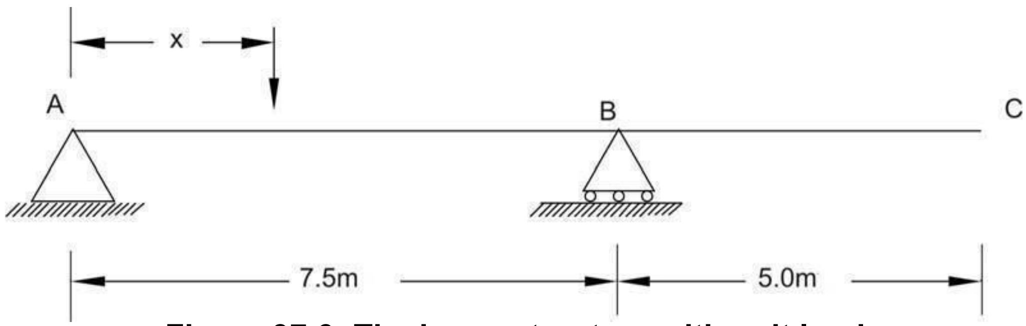


Figure 37.6: The beam structure with unit load

Similarly one can place a unit load at distances 5.0 m and 7.5 m from support A and compute reaction at B. When the load is placed at 10.0 m from support A, then reaction at B can be computed using following equation.

$$\Sigma M_A = 0 : R_B \times 7.5 - 1 \times 10.0 = 0 \Rightarrow R_B = 1.33$$

Similarly a unit load can be placed at 12.5 and the reaction at B can be computed. The values of reaction at B are tabulated as follows.

x	R_B
0	0.0
2.5	0.33
5.0	0.67
7.5	1.00
10	1.33
12.5	1.67

Graphical representation of influence line for R_B is shown in Figure 37.7.

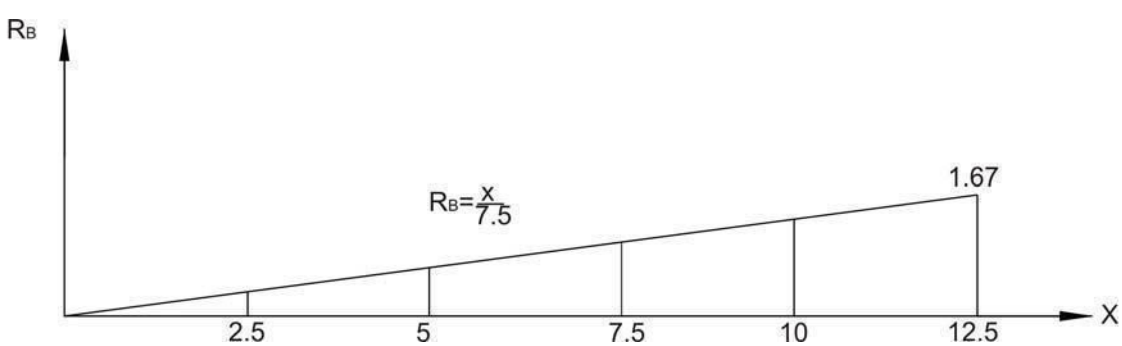


Figure 37.7: Influence for reaction R_B.

Influence line Equation:

Applying the moment equation at A (Figure 37.6),

$$\Sigma M_A = 0 : R_B \times 7.5 - 1 \times x = 0 \Rightarrow R_B = x/7.5$$

The influence line using this equation is shown in Figure 37.7.

Example 3:

Construct the influence line for shearing point C of the beam (Figure 37.8)

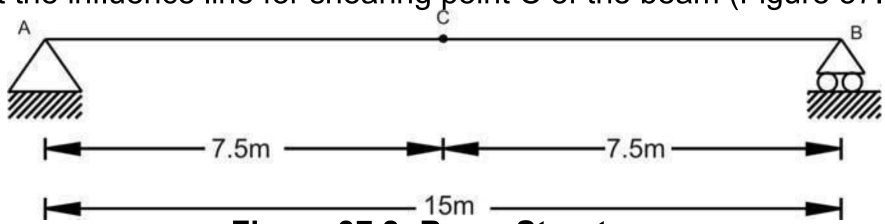


Figure 37.8: Beam Structure

Solution:

Tabulated Values:

As discussed earlier, place a unit load at different location at distance x from support A and find the reactions at A and finally computer shear force taking section at C. The shear force at C should be carefully computed when unit load is placed before point C (Figure 37.9) and after point C (Figure 37.10). The resultant values of shear force at C are tabulated as follows.

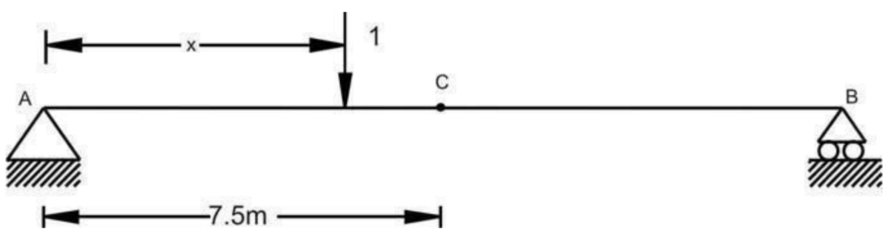


Figure 37.9: The beam structure – a unit load before section

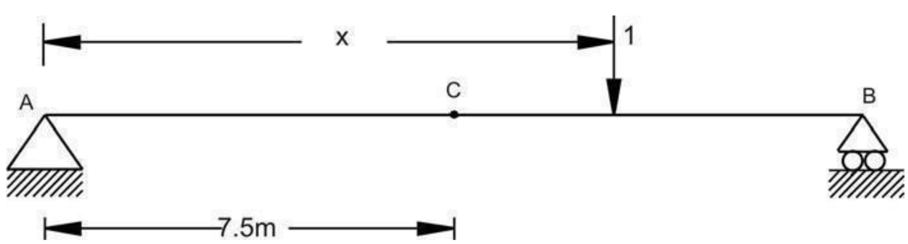


Figure 37.10: The beam structure - a unit load before section

X	V_C
0	0.0
2.5	-0.16
5.0	-0.33
7.5(-)	-0.5
7.5(+)	0.5
10	0.33
12.5	0.16
15.0	0

Graphical representation of influence line for V_C is shown in Figure 37.11.

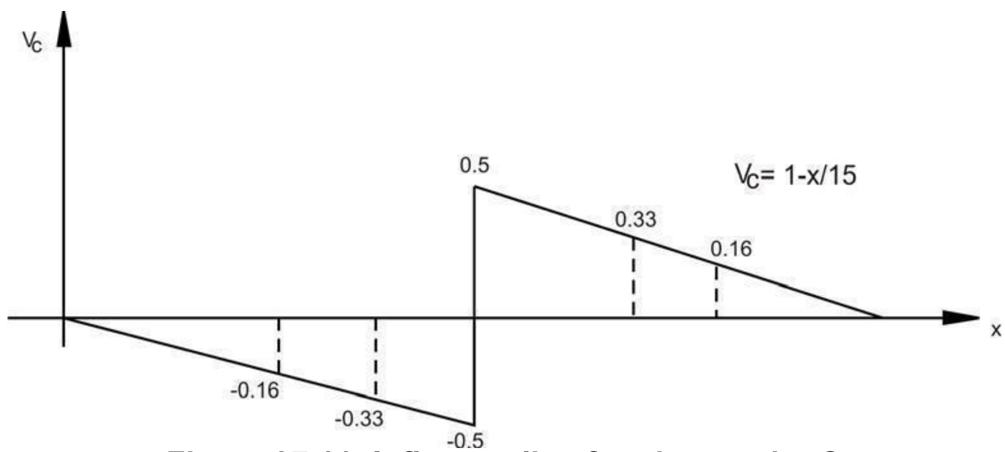


Figure 37.11: Influence line for shear point C

Influence line equation:

In this case, we need to determine two equations as the unit load position before point C (Figure 37.12) and after point C (Figure 37.13) will show different shear force sign due to discontinuity. The equations are plotted in Figure 37.11.

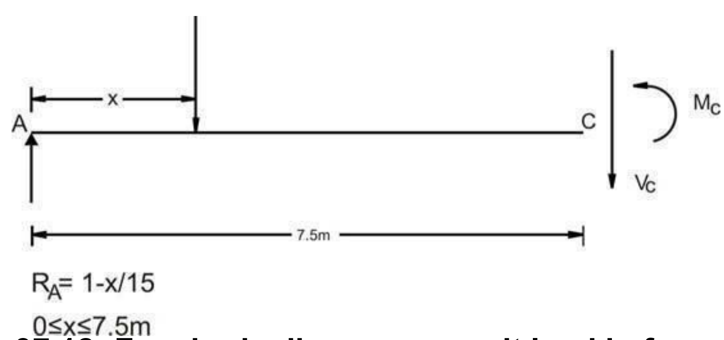


Figure 37.12: Free body diagram – a unit load before section

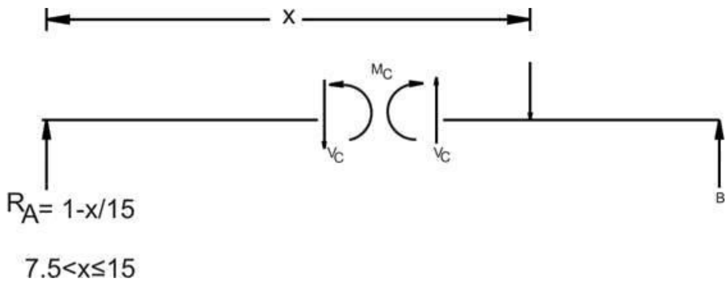


Figure 37.13: Free body diagram – a unit load after section

Influence Line for Moment:

Like shear force, we can also construct influence line for moment.

Example 4:

Construct the influence line for the moment at point C of the beam shown in Figure 37.14

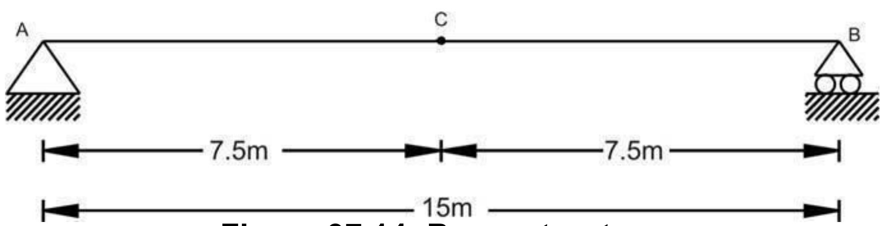


Figure 37.14: Beam structure

Solution:

Tabulated values:

Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example, we place the unit load at x=2.5 m from support A (Figure 37.15), then the support reaction at A will be 0.833 and support reaction B will be 0.167. Taking section at C and computation of moment at C can be given by

$$\Sigma M_C = 0 : - M_C + R_B \times 7.5 - = 0 \Rightarrow - M_C + 0.167 \times 7.5 - = 0 \Rightarrow M_C = 1.25$$

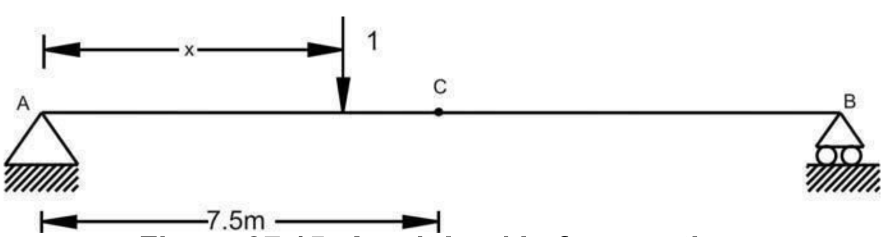


Figure 37.15: A unit load before section

Similarly, compute the moment M_C for difference unit load position in the span.

The values of Mc are tabulated as follows.

X	M_C
0	0.0
2.5	1.25
5.0	2.5
7.5	3.75
10	2.5
12.5	1.25
15.0	0

Graphical representation of influence line for M_C is shown in Figure 37.16.

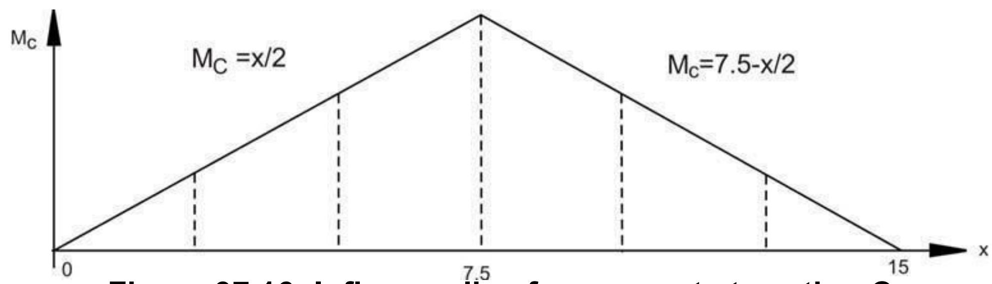


Figure 37.16: Influence line for moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When the unit load is placed before point C then the moment equation for given Figure 37.17 can be given by

$\Sigma M_C = 0 : M_C + 1(7.5 - x) - (1 - x/15)x7.5 = 0 \Rightarrow M_C = x/2$, where $0 \le x \le 7.5$

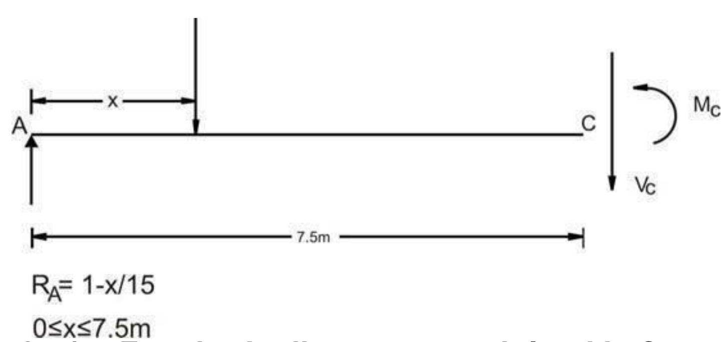


Figure 37.17: Free body diagram - a unit load before section

When the unit load is placed after point C then the moment equation for given Figure 37.18 can be given by

$\Sigma M_C = 0 : M_C - (1 - x/15) \times 7.5 = 0 \Rightarrow M_C = 7.5 - x/2$, where $7.5 < x \le 15.0$

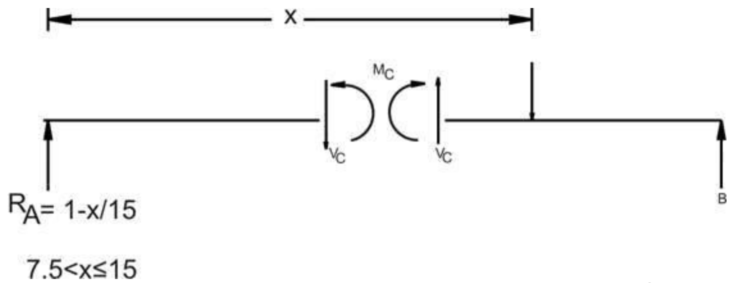


Figure 37.18: Free body diagram - a unit load before section

The equations are plotted in Figure 37.16.

Example 5:

Construct the influence line for the moment at point C of the beam shown in Figure 37.19.

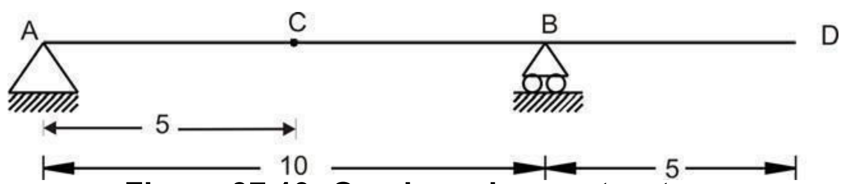


Figure 37.19: Overhang beam structure

Solution:
Tabulated values:

Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example as shown in Figure 37.20, we place a unit load at 2.5 m from support A, then the support reaction at A will be 0.75 and support reaction B will be 0 .25.

Figure 37.20: A unit load before section C

Taking section at C and computation of moment at C can be given by

$$\Sigma M_C = 0 : - M_C + R_B \times 5.0 - = 0 \Rightarrow - M_C + 0.25 \times 5.0 = 0 \Rightarrow M_C = 1.25$$

Similarly, compute the moment M_C for difference unit load position in the span. The values of Mc are tabulated as follows.

x	M_C
0	0
2.5	1.25
5.0	2.5
7.5	1.25
10	0
12.5	-1.25
15.0	-2.5

Graphical representation of influence line for M_C is shown in Figure 37.21.

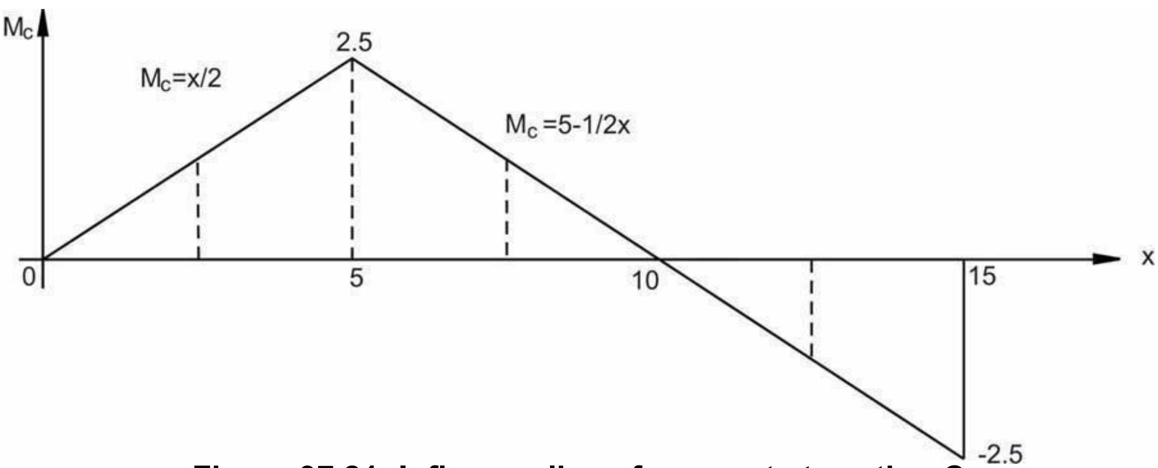


Figure 37.21: Influence line of moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When a unit load is placed before point C then the moment equation for given Figure 37.22 can be given by

$\Sigma M_C = 0 : M_C + 1(5.0 - x) - (1 - x/10) \times 5.0 = 0 \Rightarrow M_C = x/2$, where $0 \le x \le 5.0$

Figure 37.22: A unit load before section C

When a unit load is placed after point C then the moment equation for given Figure 37.23 can be given by

$\Sigma M_C = 0 : M_C - (1 - x/10) \times 5.0 = 0 \Rightarrow M_C = 5 - x/2$, where $5 < x \le 15$

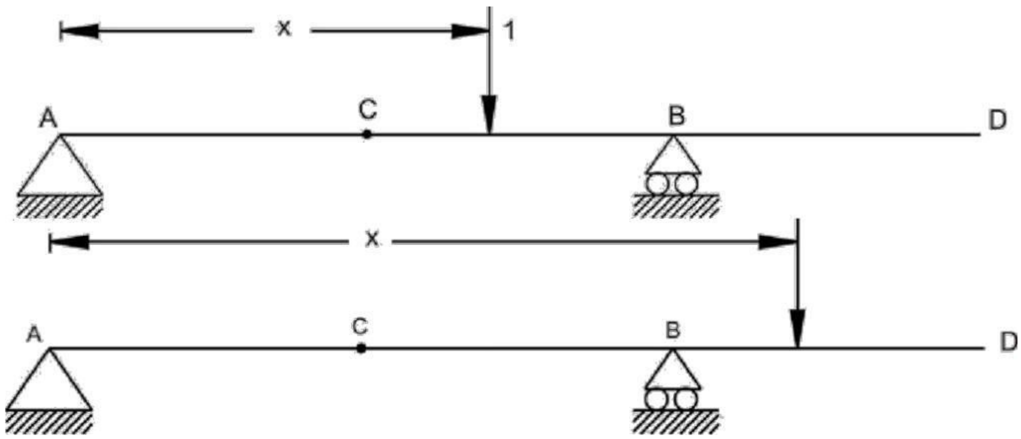


Figure 37.23: A unit load after section C

The equations are plotted in Figure 37.21.

37.5 Influence line for beam having point load and uniformly distributed load acting at the same time

Generally in beams/girders are main load carrying components in structural systems. Hence it is necessary to construct the influence line for the reaction, shear or moment at any specified point in beam to check for criticality. Let us assume that there are two kinds of load acting on the beam. They are concentrated load and uniformly distributed load (UDL).

37.5.1 Concentrated load

As shown in the Figure 37.24, let us say, point load P is moving on beam from A to B. Looking at the position, we need to find out what will be the influence line for reaction B for this load. Hence, to generalize our approach, like earlier examples, let us assume that unit load is moving from A to B and influence line for reaction A can be plotted as shown in Figure 37.25. Now we want to know, if load P is at the center of span then what will be the value of reaction A? From Figure 37.24, we can find that for the load position of P, influence line of unit load gives value of 0.5. Hence, reaction A will be 0.5xP. Similarly, for various load positions and load value, reactions A can be computed.

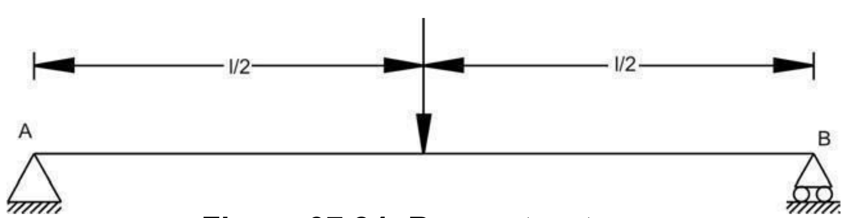


Figure 37.24: Beam structure

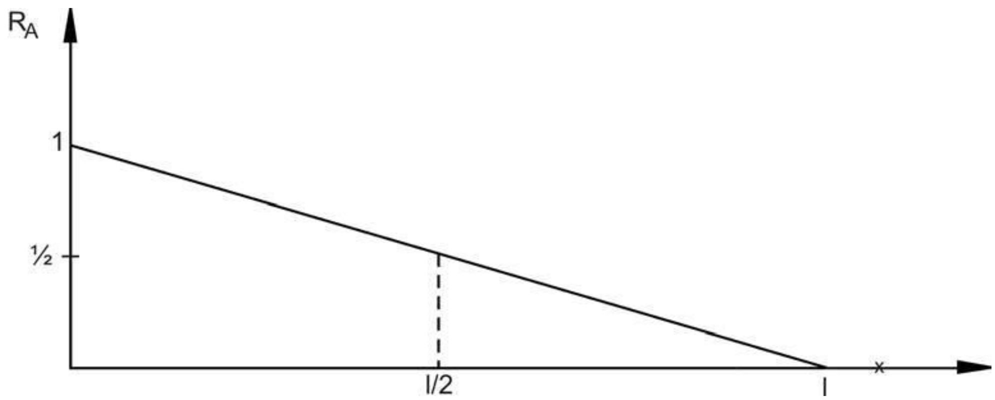


Figure 37.25: Influence line for support reaction at A

37.5.2 Uniformly Distributed Load

Beam is loaded with uniformly distributed load (UDL) and our objective is to find influence line for reaction A so that we can generalize the approach. For UDL of w on span, considering for segment of dx (Figure 37.26), the concentrated load dP can be given by w.dx acting at x. Let us assume that beam's influence line ordinate for some function (reaction, shear, moment) is y as shown in Figure 37.27. In that case, the value of function is given by (dP)(y) = (w.dx).y. For computation of the effect of all these concentrated loads, we have to integrate over the entire length of the beam. Hence, we can say that it will be ∫ w.y.dx = w ∫ y.dx. The term ∫ y.dx is equivalent to area under the influence line.

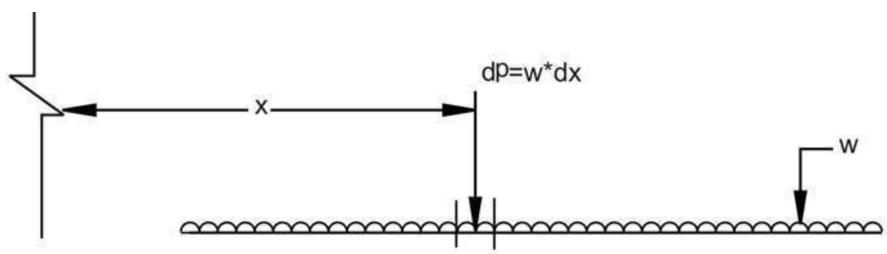


Figure 37.26: Uniformly distributed load on beam

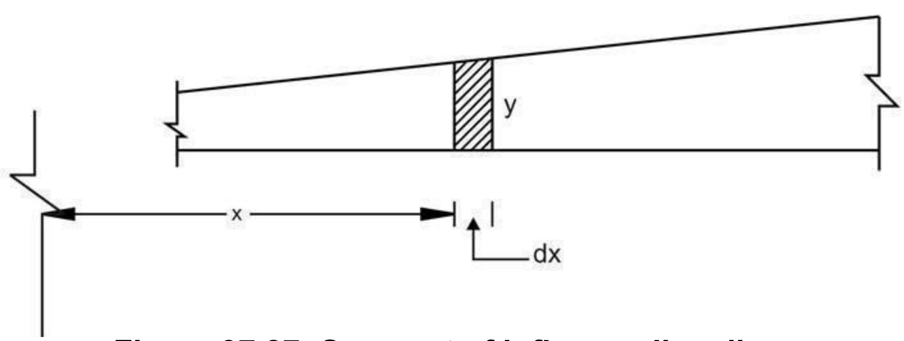


Figure 37.27: Segment of influence line diagram

For a given example of UDL on beam as shown in Figure 37.28, the influence line (Figure 37.29) for reaction A can be given by area covered by the influence line for unit load into UDL value. i.e. [0.5x (1)xl] w = 0.5 w.l.

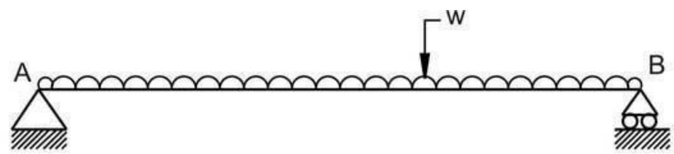


Figure 37.28: UDL on simply supported beam

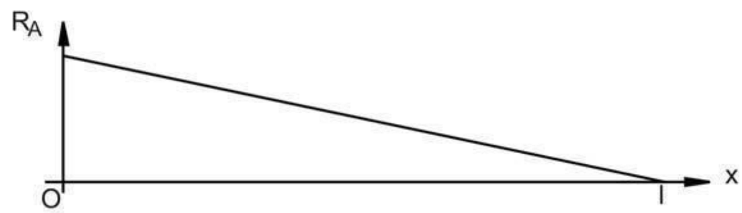


Figure 37.29: Influence line for support reaction at A.

37.6 Numerical Example

Find the maximum positive live shear at point C when the beam (Figure 37.30) is loaded with a concentrated moving load of 10 kN and UDL of 5 kN/m.

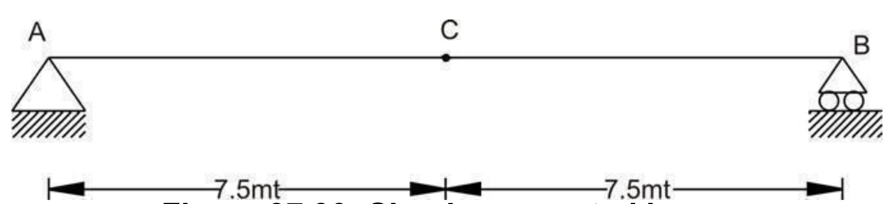


Figure 37.30: Simply supported beam

Solution:

As discussed earlier for unit load moving on beam from A to B, the influence line for the shear at C can be given by following Figure 37.31.

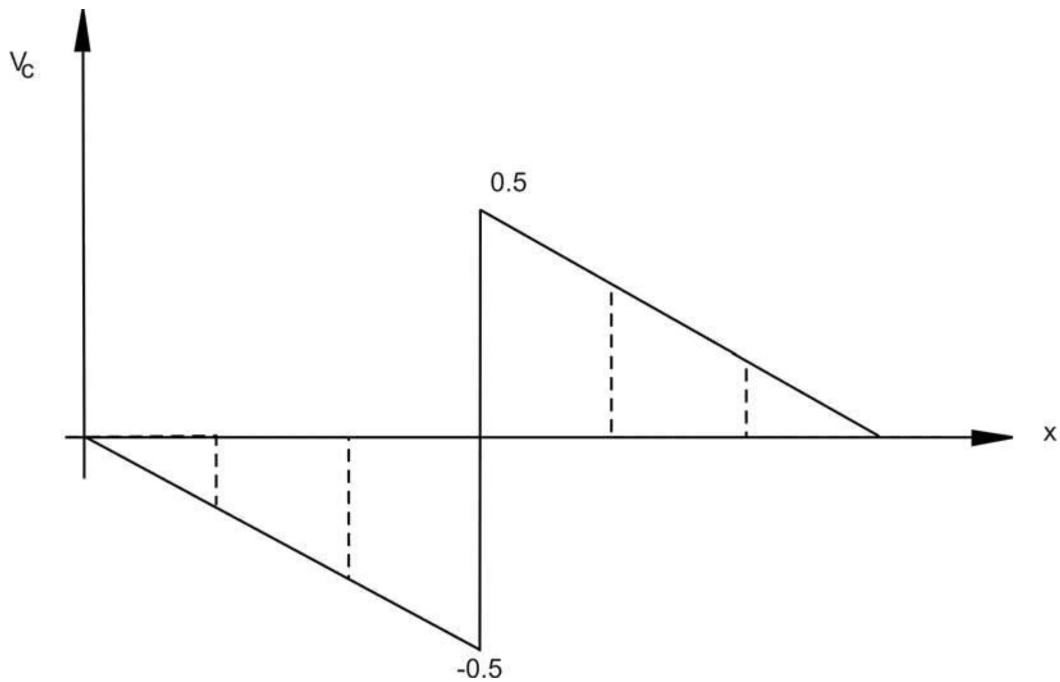


Figure 37.31: Influence line for shear at section C.

Concentrated load: As shown in Figure 37.31, the maximum live shear force at C will be when the concentrated load 10 kN is located just before C or just after C. Our aim is to find positive live shear and hence, we will put 10 kN just after C. In that case,

$$V_C = 0.5 \times 10 = 5 \text{ kN}.$$

UDL: As shown in Figure 37.31, the maximum positive live shear force at C will be when the UDL 5 kN/m is acting between x = 7.5 and x = 15.

$$V_C = [\ 0.5 \times (15 - 7.5)\ (0.5)] \times 5 = 9.375$$

Total maximum Shear at C:

$$(V_C)\ \max = 5 + 9.375 = 14.375.$$

Finally the loading positions for maximum shear at C will be as shown in Figure 37.32. For this beam one can easily compute shear at C using statics.

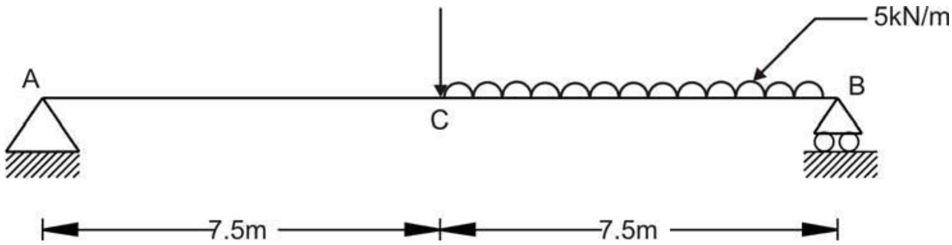


Figure 37.32: Simply supported beam

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